THE FEBRUARY MEETING IN NEW YORK

The two hundred sixty-seventh regular meeting of the Society was held at Columbia University, on Saturday, February 23, 1929, extending through the usual morning and afternoon sessions. The attendance included the following fifty-nine members.

R. L. Anderson, R. G. Archibald, H. E. Arnold, Benton, G. A. Campbell, J. A. Clark, Cramlet, Demos, Doermann, J. E. Donahue, Dresden, Engstrom, Fite, D. A. Flanders, R. M. Foster, Gehman, Gill, Glenn, Gourin, Gronwall, Hazeltine, L. S. Hill, Himwich, E. H. Johnson, M. I. Johnson, Kasner, Kenny, Kline, Koopman, Mark Kormes, Kramer, Marden, Mullins, F. H. Murray, Newman, Parsons, Pfeiffer, Pierpont, Post, Rawles, Reddick, M. S. Rees, R. G. D. Richardson, Ritt, Robertson, Seely, Serghiesco, Simons, Smail, P. A. Smith, J. M. Thomas, J. L. Walsh, Weinstein, Weisner, Whittemore, H. B. Williams, W. J. Willis, W. A. Wilson, Zippin.

There was no meeting of the Council or of the Trustees of the Society.

Associate Secretary Dresden presided at the morning session, and Professor Whittemore in the afternoon.

At the request of the Program Committee, Professor J. L. Walsh, of Harvard University, delivered, at the beginning of the afternoon session, an address entitled *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions*. This address will appear in full in an early issue of this Bulletin.

Titles and abstracts of the other papers read at this meeting follow below. The papers of Beckenbach, Calugaréano, Douglas and Emch were read by title. Mr. Beckenbach was introduced by Professor Bray, and Dr. Calugaréano by Professor Kasner.

1. Professor J. E. Donahue: *Geometric proof of Kasner's pentagon theorem.*

In a recent paper (American Mathematical Monthly, vol. 35 (1928), pp. 352 ff.), Kasner proves (using coordinates) the following theorem: *For*

any plane pentagon P the inscribed pentagon of its diagonal pentagon and the diagonal pentagon of its inscribed pentagon are identical. In the present paper there is given a synthetic proof employing Pascal's line and Brianchon's point.

2. Miss Edna E. Kramer: *The Laguerre group and allied topics.*

It is well known that the Laguerre group can be represented analytically by the linear fractional transformations of the variable $w=u+jv$, where $j^2=0$ and u and v are functions of the Hessian coordinates of a directed line. Using this representation, the present paper deals with the Laguerre group in a manner analogous to that in which the inversion group is often treated. Some new properties of Laguerre inversion are discussed, and a large family of anallagmatic curves is found. The invariants and path-curves of the groups of integral linear transformations are discussed. For $W=w+b$ these curves are cycles (directed circles); for $W=aw$ each curve (non-oriented) is a tractrix. For the direct transformations of the Laguerre group it is found that there may be 0, 2, ∞^1, or $2\infty^1$ fixed rays in the finite plane, and for the indirect transformations, 1, 2, ∞^1 or $2\infty^1$. The transformations resulting from combinations of the integral linear transformations with reciprocation are classified kinematically. In the second part of the paper a theory of polygenic functions of w is developed, analogous to that of Kasner for polygenic functions of z, and points of similarity and divergence between the two theories are noted.

3. Professor Edward Kasner: *The biharmonic functions of Poincaré related to functions of two complex variables.*

Poincaré applies the term biharmonic to those functions of four real variables $F(x, y, x', y')$ which can be regarded as the real part of an analytic function of two complex variables $z=x+iy$, $z'=x'+iy'$. If a conformal substitution $x'=\phi(x, y)$, $y'=\psi(x, y)$ is performed, F obviously becomes a harmonic function of x, y. In the present paper two easy converses of this theorem are proved. (I) Biharmonic functions are the only functions of four real variables which are converted into harmonic functions by *every* conformal substitution. (II) The only substitutions which convert *all* biharmonic functions into harmonic functions are the conformal substitutions. Connection is made with the author's geometric representation of functions of two complex variables, first by points in four dimensions, then by bipoints or point-pairs in the plane. See abstracts in this Bulletin, vol. 15 (1908–09), pp. 67, 159.

4. Mr. Eli Gourin: *On the irreducible polynomials in several variables which become reducible when the variables are replaced by powers of themselves.*

Given an irreducible polynomial $Q(x_1, \cdots, x_p)$, consisting of more than two terms, the sets of positive integers $l_1, \cdots, l_p$ for which $Q(x_1^{l_1}, \cdots, x_p^{l_p})$ is reducible can be grouped into a finite number of classes of sets. In each class, the sets are of the form $a_1T_1, \cdots, a_pT_p$, where the T's are elements of a certain set, having the required property, and the a's are arbitrary

positive integers. Those of the T's which are distinct from unity are each equal to one and the same prime number p. If M denotes the largest among the m_i's, where m_i is the degree of Q in x_i, and if P is the largest prime number which does not exceed M^2, then $p \leqq P$.

5. Professor J. F. Ritt: *Algebraic combinations of exponentials.*

This paper investigates functions defined by an equation

$$(1) \qquad \sum_{i=0}^{n} (a_{0i}+a_{1i}e^{\alpha_1 z}+\cdots+a_{mi}e^{\alpha_m z})y^i = 0,$$

where the α's and a's are any complex constants. The theory of these functions contains in itself the theory of algebraic functions. It is shown that, for large x's, every branch of a function defined by (1) can be represented by Dirichlet series with complex exponents. It is then proved that if (1) defines more than a single analytic function, the first member of (1) can be factored into expressions similar to itself but of lower degree in y. In particular, a uniform function defined by (1) must satisfy an equation of the first degree, and hence is the quotient of two exponential polynomials.

6. Professor J. F. Ritt: *Representation of analytic functions as infinite products.*

Let $f(z)$ be analytic, and equal to unity, at $z=0$. It is proved in this paper that $f(z)$ admits, for the neighborhood of the origin, a representation as an absolutely convergent product $(1+a_1z)(1+a_2z^2)\cdots(1+a_nz^n)\cdots$, with constant a's. The representation is unique.

7. Professor W. A. Wilson: *On linear upper semi-continuous collections of continua.*

Let M be a bounded upper semi-continuous collection of continua $\{X\}$ such that no two have common points and $X=f(t)$ is upper semi-continuous in the interval $a \leqq t \leqq b$; a collection of this kind may be called *linear* and M the *image* of ab. If there is no upper semi-continuous aggregate function $Y=g(t)$, where each Y is a continuum defined in ab such that $g(t) \subset f(t)$ at every point and $g(t) \neq f(t)$ at some point, $f(t)$ is called a minimal upper semi-continuous function. It was shown in a previous communication that, if $f(t)$ is a minimal upper semi-continuous function in ab, M is a bounded plane continuum, and no $f(t)$ separates $f(a)$ from $f(b)$, then M is irreducible between $f(a)$ and $f(b)$. The present paper discusses the case that the set of points $T'=\{t'\}$ for which $f(t')$ separates $f(a)$ from $f(b)$ is not void. It is shown that the conclusion of the quoted theorem remains valid if the set T' is totally disconnected, and that it is sometimes valid and sometimes not if T' contains an interval. Finally, there exist cases where $f(t)$ is a minimal upper semi-continuous function defined in an interval ab and the image of ab fills a portion of the plane.

8. Dr. O. E. Glenn: *A theory of integral invariants.*

The general question studied in this paper is as follows. A curve C: $f(x, y)=0$, is transformed by $x_1=\sigma(x, y)$, $y_1=\tau(x, y)$ into C_1: $g(x_1, y_1)$.

Suppose any function ϕ of x, y, y' to be given. It is required to find the line of integration C such that $\int_\alpha^\beta \phi(x_1, y_1, y_1')dx = M\int_\gamma^\delta \phi(x, y, y')dx$, the line of integration being C_1 in the first integral and C in the second. The limits (α, β) are to be arbitrary, and (γ, δ) are limits transformed from (α, β) by $x_1=\sigma$, $y_1=\tau$. We also consider the problem where ϕ is to be determined, C being assigned.

9. Dr. G. Calugaréano: *Polygenic functions considered as integrals of differential equations.*

In this paper, which will appear in the Transactions of this Society, the author applies polygenic (that is, non-analytic) functions to the solution of certain ordinary differential equations of nth order. The first derivative is defined as in Kasner's papers (Science, vol. 66 (1927), p. 981; Proceedings of the National Academy, vol. 13 (1928, p. 75). The second and higher derivatives are the special case which Kasner describes as rectilinear (Transactions of this Society, vol. 30 (1928), pp. 803–818). By means of the polygenic solutions, analytic solutions may also be obtained. The discussion is purely analytic, as in the author's Paris thesis (Nov., 1928).

10. Professor Arnold Emch: *On the mapping of the involutorial G_4 in a plane upon a Steiner surface.*

This paper appears in full in the present issue of this Bulletin.

11. Dr. Jesse Douglas (National Research Fellow): *Solution of the problem of Plateau.*

In a paper presented to the Society, Dec. 28, 1926, the author reduced the problem of Plateau to the integral equation (I): $*\int_\Gamma K(t, \tau)$ ctn $(\frac{1}{2}[\phi(\tau) - \phi(t)])d\tau = 0$ (or its equivalent by the substitution ctn $(\frac{1}{2}\phi) = \psi$: $*\int_\Gamma K(t, \tau)d\tau/[\phi(\tau) - \phi(t)] = 0$). For the notation see the abstract in this Bulletin, vol. 33, p. 143. In a subsequent paper presented April 6, 1928 (see this Bulletin, vol. 34, p. 405) the author showed that (I) expressed the vanishing of the functional derivatives of the functional

$$A(\phi) = -\int_\Gamma\int_\Gamma K(t, \tau) \log \sin \tfrac{1}{2}|(\phi(\tau) - \phi(t)|\, dtd\tau.$$

It thus becomes a question of proving the existence of a minimizing function ϕ for $A(\phi)$. This is accomplished in the present paper. It is shown that $A(\phi)$ is a lower semi-continuous function on the set of all functions ϕ which, by means of $\theta = \phi(t)$, establish a one-one continuous correspondence between the contour Γ and the unit circle C; only these functions are admissible. In order to have a closed set $[\phi]$ it is necessary to include functions ϕ whose cartesian t, θ graph presents (1) horizontal or (2) vertical line segments as well as (3) properly monotonically increasing continuous functions ϕ. The set $[\phi]$ is compact as well as closed, and we can therefore apply the results of Fréchet's thesis (Palermo Rendiconti, vol. 22 (1906)) to affirm the existence of a minimizing ϕ. It is shown that, for a function ϕ of type (1), $A(\phi) = +\infty$, and for one of type (2), $A'(\phi, t)$ cannot be identically zero. Hence the existing minimizing ϕ is of the proper type (3), and the problem is solved.

12. Mr. E. F. Beckenbach: *An inequality for definite Hermitian determinants.*

This paper appears in full in the present issue of this Bulletin.

13. Dr. T. H. Gronwall: *Straight line geodesics in Einstein's parallelism geometry.* Preliminary communication.

This paper gives a set of partial differential equations forming the necessary and sufficient conditions that, in Einstein's parallelism geometry, all geodesics be straight lines and vice versa.

14. Dr. T. H. Rawles: *On the inverse problem in the calculus of variations.*

The problem considered in this paper is that of finding the integrand function which is minimized by a given two-parameter family of extremals. By approaching the problem from the standpoint of the invariant integral a method is obtained which is simpler than that of Darboux in that it is not necessary to solve a partial differential equation.

15. Professor James Pierpont: *Foucault's pendulum in elliptic space.*

The equations of motion are deduced from Hamilton's principle. We find $2\dot{\psi}(\dot{\omega}+kp)+\psi\ddot{\omega}=k^2p'\psi \sin\omega\cos\omega-k^2pp'(r/l)\sin\omega$, where ω is the azimuth of the plane of vibration, ψ the small angle that the pendulum makes with the vertical, k the angular velocity of the earth, ϕ the latitude of the station, $p=\sin\phi$, $p'=\cos\phi$, $l=\sin L$, $r=\sin\rho$, L being the length of the pendulum, ρ being the distance of the point of suspension from the center of the earth, both expressed in elliptic measure. The space constant is taken as unity. The above equation is entirely analogous to the corresponding equation in classical mechanics. Under similar conditions we may say, therefore, that in first approximation the angular velocity of the plane of vibration is $\dot{\omega}=-k\sin\phi$. The author wishes to note that the analysis employed may be readily extended to hyperbolic space.

16. Dr. T. H. Gronwall: *On the differential equation of the vibrating membrane.*

This paper considers the question of the analytic continuation of a solution of the partial differential equation $\nabla^2u+k^2u=0$ (with boundary condition $u=0$ or $\partial u/\partial n=0$) across a straight line part of the boundary. The result is applied to the determination of the frequencies and characteristic functions for a rectangular membrane.

17. Mr. Leo Zippin: *A characterization of the simple closed surface and of the plane.*

It is proved that a continuous curve C, such that (i) any (1, 1) bicontinuous transformation of a simple closed curve of C into itself or another simple closed curve of C can be extended to C, and (ii) there is at least one simple closed curve of C which disconnects it, is a simple closed surface if C is bounded and homeomorphic with the plane if C is unbounded.

R. G. D. RICHARDSON,
Secretary.

MODERN HYDRODYNAMICAL THEORY, WITH SPECIAL REFERENCE TO AERONAUTICS*

BY F. D. MURNAGHAN

A. General Notions and Theorems

1. *Particle and Local Differentiation.* In hydrodynamical theory matter is regarded as distributed continuously, but its various points are supposed to be capable of identification; we shall refer to them as particles. In order to identify a particle, a material reference frame is introduced, and we shall suppose for simplicity that this is a rectangular Cartesian frame. At any convenient initial instant $t=t_0$, the coordinates of a particle may be denoted by (a, b, c); and at any later instant the coordinates of the same particle may be denoted by (x, y, z). We make the assumption that (x, y, z) are differentiable functions of a, b, c, and t. This assumption

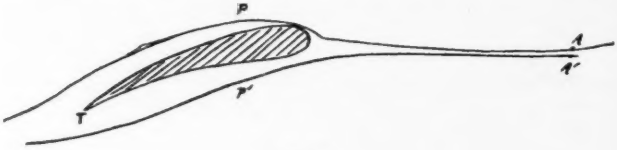

Fig. 1

must be carefully noted since the principal object of interest to us later on will be the flow of a fluid around an obstacle and the fluid will divide at the obstacle. Thus two particles initially adjacent (as A and A' in Fig. 1) will not be adjacent when separated by the obstacle as at P and P'. Our subse-

* This paper is a somewhat formal and detailed presentation of the contents of an address delivered at the invitation of the program committee at the meeting of this Society on October 29, 1927. The adjective modern in the title of the address refers to the first quarter of the present century, but in order to make the paper easily intelligible to mathematical readers who have not specialized in hydrodynamics, I give a statement of the more important results which had been previously obtained.

quent remarks will apply, then, separately to the two portions of the fluid which separate at the obstacle but not to the fluid as a whole. A particle may be identified either by means of (a, b, c, t) which are known as the particle, or Lagrangian, variables or by means of (x, y, z, t) which are known as the local, or Eulerian, variables; we shall denote time differentiation when the particle variables are used by d/dt and when the local variables are used by $\partial/\partial t$. The velocity components $(dx/dt, dy/dt, dz/dt)$ being denoted by (u, v, w), we have the relation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{1}$$

In particular the acceleration components $(du/dt, dv/dt, dw/dt)$ are given, in terms of the local variables, by formulas of the type

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}. \tag{2}$$

The paths of the various particles are found, if (u, v, w) are known in terms of the local variables, by integrating the *three* equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt. \tag{3}$$

At any given instant $t = t_1$ the *stream-lines* are given by integrating the *two* equations

$$\frac{\delta x}{u_1} = \frac{\delta y}{v_1} = \frac{\delta z}{w_1}, \tag{4}$$

where $u_1 = u(x, y, z, t_1)$, and so on. The ∞^3 paths must then be distinguished from the ∞^2 stream-lines. They coincide geometrically when the stream-lines are permanent, that is, when $\partial u/\partial t = 0$, $\partial v/\partial t = 0$, $\partial w/\partial t = 0$ identically in t. In this case the motion is said to be *steady*, and in the applications with which we shall be concerned steady motion is assumed.

2. *The Principle of Conservation of Mass.* This postulates the existence of a positive density function ρ, which we assume to be differentiable, and such that the volume integral $\int\rho\ \delta(x, y, z)$ does not vary with t, the volume of integration consisting of the same particles. In other words $\int\rho\ \delta(x, y, z)$ is an integral invariant of the equations (3). If the initial value of ρ be denoted by ρ_0, the differential form of statement of the principle of conservation is, in terms of the particle variables,

$$\rho\frac{d(x,y,z)}{d(a,b,c)} = \rho_0,$$

and in terms of the local variables,

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0. \tag{5}$$

This equation is known as the equation of continuity and it may be put in the equivalent form

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0.$$

For an incompressible fluid $d\rho/dt=0$ and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{6}$$

3. *The Circulation Integral.* With any closed curve C of particles we may associate, at any instant t, the number

$$\Gamma=\oint(u\delta x+v\delta y+w\delta z), \tag{7}$$

where δ indicates the differential with respect to the parameter of the curve. The integral Γ is known as the circulation around the curve; and an application of Stokes' theorem enables us to write it in the equivalent form

$$\Gamma = \int \xi\,\delta(y,z) + \eta\,\delta(z,x) + \zeta\,\delta(x,y), \tag{8}$$

where

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

are the vorticity components and the surface of integration is bounded by C. A simple calculation shows that

$$\frac{d\Gamma}{dt} = \oint \left\{ \left(\frac{\partial}{\partial y}\frac{dw}{dt} - \frac{\partial}{\partial z}\frac{dv}{dt} \right) \delta(y,z) \right. \tag{9}$$

$$\left. + \left(\frac{\partial}{\partial z}\frac{du}{dt} - \frac{\partial}{\partial x}\frac{dw}{dt} \right) \delta(z,x) + \left(\frac{\partial}{\partial x}\frac{dv}{dt} - \frac{\partial}{\partial y}\frac{du}{dt} \right) \delta(x,y) \right\}.$$

Hence the necessary and sufficient conditions that $d\Gamma/dt = 0$ for every closed curve C are

$$\frac{\partial}{\partial y}\frac{dw}{dt} - \frac{\partial}{\partial z}\frac{dv}{dt} = 0,$$

and so forth. In other words, Γ is a (relative) integral invariant of the equations (3) provided that the acceleration components are derivable from a potential function Q, that is, provided that

$$\frac{du}{dt} = -\frac{\partial Q}{\partial x}, \quad \frac{dv}{dt} = -\frac{\partial Q}{\partial y}, \quad \frac{dw}{dt} = -\frac{\partial Q}{\partial z}, \tag{10}$$

and conversely. In our applications this will be the situation. A particular instance of importance occurs when $\Gamma = 0$ throughout a region of the fluid; then the vorticity components (ξ, η, ζ) are zero and the motion is said to be *irrotational* in that region. The velocity components are derivable from a single function ϕ as follows:

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}; \tag{11}$$

and ϕ is called the velocity potential.

4. *The Fundamental Equations and the Concept of a Perfect Fluid.* If we focus our attention on a particular point (x, y, z) of the fluid, the orientation of a particular element of area δS through (x, y, z) may be specified by means of the direction cosines (l, m, n) of one of the two directed normals to δS and we may call the side of δS towards which the normal is drawn its positive side. Then the fluid on the negative side of δS exerts on the fluid on the positive side a force whose components may be denoted by $(P_x\,\delta S,\ P_y\,\delta S,\ P_z\,\delta S)$, where (P_x, P_y, P_z) are given in terms of (l, m, n) and nine coefficients $(p_{xx}, p_{xy}, \cdots, p_{zz})$ by the equations

$$\begin{aligned} P_x &= lp_{xx} + mp_{yx} + np_{zx}, \\ P_y &= lp_{xy} + mp_{yy} + np_{zy}, \\ P_z &= lp_{xz} + mp_{yz} + np_{zz}. \end{aligned} \tag{12}$$

The nine coefficients $(p_{xx}, p_{xy}, \cdots, p_{zz})$ constitute what is called the stress or pressure tensor. If the fluid is subjected to a system of forces (X, Y, Z) per unit mass, a simple application of the fact that the total force acting on any portion of the fluid is equal to the time rate of change of its linear momentum yields the equations

$$\rho\frac{du}{dt} = \rho X - \left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right); \tag{13}$$

$$\rho\frac{dv}{dt} = \qquad ; \ \rho\frac{dw}{dt} = \qquad .$$

In deriving these we have used the fact that if there are surface forces, whose components are (F, G, H) per unit area, acting on the fluid then (see (1)) we have

$$F + lp_{xx} + mp_{yx} + np_{zx} = 0\ ;\ G + \quad = 0\ ;\ H + \quad = 0\ ; \tag{14}$$

where (l, m, n) are the direction cosines of the outward drawn normal. The theorem that the moment of the forces acting on the fluid about any axis is equal to the time rate of change of its angular momentum about this axis yields the fact that the pressure tensor is symmetric, that is,

$$(15) \qquad p_{xy} = p_{yx}, \qquad p_{zx} = p_{xz}, \qquad p_{yz} = p_{zy}.$$

A perfect fluid is by definition one such that the stress across any element is normal to it; it is this quality of inability to withstand tangential stresses that distinguishes fluids from solids. The equations (12) combined with $P_x : P_y : P_z = l : m : n$ for arbitrary values of the ratios $l : m : n$ show that the pressure tensor for a perfect fluid is the *scalar* tensor $(p, 0, 0; 0, p, 0; 0, 0, p)$, where p is the common value of p_{xx}, p_{yy}, p_{zz}. The number p is called the *pressure* of the perfect fluid and it is well to remember that the concept of pressure, as here defined, does not exist for a non-perfect or *viscous* fluid. For a perfect fluid the equations of motion are

$$(16) \qquad \rho\frac{du}{dt} = \rho X - \frac{\partial p}{\partial x}, \quad \rho\frac{dv}{dt} = \rho Y - \frac{\partial p}{\partial y}, \quad \rho\frac{dw}{dt} = \rho Z - \frac{\partial p}{\partial z},$$

to which must be added the boundary conditions

$$(17) \qquad F + lp = 0, \quad G + mp = 0, \quad H + np = 0,$$

where p denotes, in these latter equations, the pressure at the boundary of the fluid. It is clear from (16) that an acceleration potential Q exists for a perfect fluid if (1) (X, Y, Z) are derivable from a potential function V, that is, if

$$X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z},$$

and (2) if

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial P}{\partial x}, \quad \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial P}{\partial y}, \quad \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial P}{\partial z},$$

where P is any function of x, y, z and t. Then $Q = V + P$. We shall be concerned mainly with the case where the fluid is *homogeneous*, that is, $\partial\rho/\partial x = 0$, $\partial\rho/\partial y = 0$, $\partial\rho/\partial z = 0$, and in this case $P = p/\rho$ so that

$$(18) \qquad Q = V + \frac{p}{\rho}.$$

If our first condition is satisfied but not our second we have from (9)

$$\frac{d\Gamma}{dt} = \int\Bigg[\frac{\partial\left(p,\frac{1}{\rho}\right)}{\partial(y,z)}\delta(y,z) + \frac{\partial\left(p,\frac{1}{\rho}\right)}{\partial(z,x)}\delta(z,x) + \frac{\partial\left(p,\frac{1}{\rho}\right)}{\partial(x,y)}\delta(x,y)\Bigg] \tag{19}$$

$$= \int\delta\left(p,\frac{1}{\rho}\right) = \oint_C p\delta\left(\frac{1}{\rho}\right).$$

5. *Bernoulli's Theorem connecting Pressure and Velocity.* When an acceleration potential Q exists the three equations $du/dt = -\partial Q/\partial x$, etc., may be written in the form

$$\frac{\partial u}{\partial t} + \eta w - \zeta v = -\frac{\partial}{\partial x}\left(Q + \frac{1}{2}q^2\right),$$

and so forth, where (ξ, η, ζ) are the vorticity components and $q^2 = u^2+v^2+w^2$ is the square of the velocity. These yield at once the equation

$$\frac{d}{dt}\left(Q + \frac{1}{2}q^2\right) = \frac{\partial Q}{\partial t},$$

so that when Q is free of t, $Q+q^2/2$ is a first integral of the equations (3). In other words $Q+q^2/2$ has a constant value along any one path but it will, in general, vary from path to path. If, however, the motion is *steady* ($\partial u/\partial t = 0$, etc.), and *irrotational* ($\xi = 0$, etc.), we have $\partial(Q+q^2/2)/\partial x = 0$, etc., so that $Q+q^2/2$, being free of t, is a number independent of x, y, z and t. In the application with which we shall be concerned we shall have $V = 0$, $P = p/\rho$, so that $Q = V+P = p/\rho$, with ρ a numerical constant. Hence we may write

$$p + \tfrac{1}{2}\rho q^2 = p_0\,; \quad \text{or} \quad p = p_0 - \tfrac{1}{2}\rho q^2, \tag{20}$$

where p_0 is a constant throughout the fluid. Since p_0 is

what p becomes wherever $q=0$, it is called the *static* pressure. The additional term $-\rho q^2/2$ is called the *dynamic* pressure.

6. *The Forces and Turning Moments Experienced by Solids in the Fluid.* The equations (16) may be written in the form

$$(21)\quad \rho X - \frac{\partial p}{\partial x} = \rho\frac{du}{dt} = \rho\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw),$$

and so forth, on using (5). Upon integrating this equation throughout the region of the fluid between the immersed solid, or solids, S and any convenient outer boundary surface S', and using the fact that at any point of the surface of a solid the velocity of the fluid is tangential to the surface, we find that the force components (F_x, F_y, F_z) exerted by the fluid on the immersed solids are given by equations of the form

$$(22)\quad F_x = \int \rho X d\tau - \int \frac{\partial}{\partial t}(\rho u) d\tau - \int \rho u(lu + mv + nw) dS' - \int p l dS',$$

where (l, m, n) are the direction cosines of the normal to dS', drawn away from the fluid. In the cases to which we shall apply this result we shall assume the applied mass forces negligible ($X=0$, etc.), the motion steady ($\partial u/\partial t = 0$, etc.) and irrotational (so that (20) is applicable) and the fluid is compressible and homogeneous; we find then

$$(23)\quad F_x = \frac{\rho}{2}\int \{l(u^2 + v^2 + w^2) - 2u(lu + mv + nw)\} dS';$$

$$F_y = \qquad ; \quad F_z = \qquad .$$

In a similar way, the moments (M_x, M_y, M_z) about the coordinate axes of the forces exerted by the fluid on the immersed obstacle are given by equations of the form

$$(24)\quad M_x = \frac{\rho}{2}\int \{(yn - zm)(u^2 + v^2 + w^2) - 2(yw - zv)(lu + mv + nw)\} dS',$$

and so forth. It is with these forces and moments that we shall be mainly concerned and the significance of the formulas given lies in the fact that the desired values are obtained by integration over an *arbitrary* surface S' enclosing the obstacles. It will be usually convenient, when the fluid is supposed infinite in extent, to regard S' as receding to infinity and we see that a knowledge of the nature of the flow at infinity suffices to determine the forces exerted on the solids by the fluid and also the moments of these forces.

B. Two-Dimensional Irrotational Motion

7. *The Complex Velocity and Potential.* In many applications we may assume, at any rate to a first approximation, that the velocity of each particle is parallel to a given plane and independent of its distance from it. Taking the given plane as our (x, y) plane we have $w=0$ and (y, v) independent of z. We shall find it convenient to introduce the complex variable $x+iy$ and as we shall not have occasion to use the symbol z in this section in its previous sense we shall denote $x+iy$ by z. Assuming the fluid to be *incompressible* we have, from (6), the relation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{25}$$

and if we further assume the motion to be irrotational we have

$$\zeta \equiv \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0. \tag{26}$$

The equations (25) and (26) are equivalent to the statement that $u-iv$ is a function of z. We shall denote this function by $w(z)$, since there is no possibility of confusion with the third velocity component which is now everywhere zero; $w(z)$ is called the *complex velocity*. Upon introducing the function $f(z)$ defined, save for an additive constant, by the equation

$$\frac{df}{dz} = w(z), \tag{27}$$

and denoting the real and imaginary parts of f by ϕ and ψ, respectively, we have, from (27),

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{28}$$

Hence ϕ is the velocity potential and the level curves $\psi = $ constant are the stream-lines. The function f is called the complex potential. The equations (28) tell us that the resolved part v_s of the velocity vector in any direction s is given by

$$v_s = \frac{\partial \phi}{\partial s} = \frac{\partial \psi}{\partial \left(s + \frac{\pi}{2}\right)}. \tag{29}$$

Once the complex potential f is known the velocity components (u, v) follow upon differentiation and from these the pressure at any point may be calculated by means of (20), the motion being assumed to be steady.

8. *The Flow around a Circular Cylinder.* We shall suppose that the cylinder, of radius a and with generators perpendicular to the (x, y) plane, has a uniform velocity of translation V making an angle $-\alpha$ with the positive x axis. The flow relative to the cylinder will, therefore, have a velocity V making an angle $\pi - \alpha$ with the positive x axis at points remote from the cylindrical obstacle. In other words the complex velocity w will have the value $Ve^{i(\pi+\alpha)} = -Ve^{i\alpha}$ at $z = \infty$ so that, at $z = \infty$, f will have the form $-Ve^{i\alpha}z$ + terms whose derivatives vanish at $z = \infty$. Since the circle in which the cylinder intersects the (x, y) plane must be a portion of a stream line, ψ must be constant along it and this constant may be taken to be zero since f is undefined to the extent of an additive constant; hence the correspondence set up between points in the z plane and points of the

$f=\phi+i\psi$ plane is such that points on the circle in the z plane correspond to points on the axis of reals in the f plane. If the center of the circle in the z plane be denoted by z_0 and if $z_0 \pm ae^{i\delta}$ are two diametrically opposite points on the circle the relation

$$\frac{f-f_0}{f+f_0}=C\left(\frac{z-z_0-ae^{i\delta}}{z-z_0+ae^{i\delta}}\right)^2$$

will set up a correspondence of the kind desired, the ends of the segment of the real axis in the f plane being denoted by $\pm f_0$ and C being a constant. In order that f may have the proper form, $f=-Ve^{i\alpha}z+\cdots$, at ∞, it is necessary that $C=1$ and $\delta=\pi-\alpha$, $f_0=2aV$. The terms in f whose derivatives vanish at $z=\infty$ may include, since f need not be uniform owing to the fact that the portion of the z plane outside the circle is not simply connected, a term of the form $A \log (z-z_0)$; for if A is a pure imaginary, the imaginary part of $A \log (z-z_0)$ is constant as we go along the circle in the z plane. If we write $A=-i\Gamma/(2\pi)$, the quantity Γ will be the circulation around any closed curve in the (x, y) plane which enlaces the cylinder once and we have

$$(30)\quad f=-V\left[(z-z_0)e^{i\alpha}+\frac{a^2e^{-i\alpha}}{z-z_0}\right]-\frac{i\Gamma}{2\pi}\log(z-z_0).$$

We shall see in the next paragraph how Γ is determined for obstacles with a sharp trailing edge. It was precisely this determination of Γ which constituted the first significant result belonging to what we have termed the modern period.

On differentiating f it is seen that there are two points where w is zero. If $\Gamma \leqq 4\pi aV$, these two points are on the circle and are symmetrically situated with respect to the diameter of the circle which is perpendicular to the direction of flight. The two points are known as *rest points* and they are distinguished from one another by the terms *front* and *rear*. If the argument of the rear rest-point be denoted by $\pi+\beta$ we have

$$\Gamma = 4\pi a V \sin(\alpha + \beta). \tag{31}$$

9. *Flow around an Airfoil.* If we have an obstacle in the z plane which can be mapped on a circle in an auxiliary ζ plane by means of a relationship of the type

$$\zeta = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \tag{32}$$

which makes the points at ∞ in the two planes correspond to one another, it is clear from (30) that the desired complex potential is given by

$$f = -V\left[(\zeta - \zeta_0)e^{i\alpha} + \frac{a^2 e^{-i\alpha}}{\zeta - \zeta_0}\right] - \frac{i\Gamma}{2\pi}\log(\zeta - \zeta_0), \tag{33}$$

where ζ is defined by (32), a is the radius, and ζ_0 the center of the circle in the auxiliary plane. We shall be interested mainly in obstacles whose contour is of the type illustrated in Fig. 1; these have a cusp or sharp double point at the trailing edge T and so $dz/d\zeta$ will be zero there. Now the complex velocity

$$w = \frac{df}{dz} = \frac{df}{d\zeta} : \frac{dz}{d\zeta},$$

and this will be infinite at the trailing edge unless $df/d\zeta$ is zero there. This condition suffices to determine the unknown circulation Γ. Physically it amounts to saying that for a stream-lined airfoil the flow is smooth at the trailing edge. Geometrically stated, the rear rest-point on the circle in the auxiliary ζ plane must be mapped on the trailing edge of the obstacle in the z plane.

10. *The Blasius Formulas for Lift and Turning-Moment.* If ds' denote the element of arc of any curve C' enclosing the contour of the obstacle, the components of the force *per unit length* exerted on the obstacle by the fluid are, from (23),

$$F_x = \frac{\rho}{2} \oint_{C'} \{l(u^2 + v^2) - 2u(lu + mv)\} ds',$$

$$F_y = \frac{\rho}{2} \oint_{C'} \{m(u^2 + v^2) - 2v(lu + mv)\} ds',$$

where (l, m) are the (x, y) direction cosines of the outward drawn normal to C'. Since $l = dy'/ds'$ and $m = -dx'/ds'$, we readily find that

$$\overline{F} \equiv F_x - iF_y = \frac{i\rho}{2} \oint_{C'} w^2 dz. \tag{34}$$

This important formula, due to Blasius, enables us to write down at once F_x and F_y when the development of w^2 near $z = \infty$ is known.

Similarly the turning moment M, about an axis through the origin perpendicular to the (x, y) plane, of the forces exerted by the fluid on a unit length of the cylinder, being equal to

$$\frac{\rho}{2} \oint_{C'} \{(xm - yl)(u^2 + v^2) - 2(xv - yu)(lu + mv)\} ds',$$

is seen to be the real part of

$$-\frac{\rho}{2} \oint_{C'} zw^2 dz,$$

that is,

$$M = \text{R.P.} \left(-\frac{\rho}{2} \oint_{C'} zw^2 dz\right). \tag{35}$$

In order to apply these results we turn to (32) and (33). Thus, from

$$\zeta = z\left(1 + \frac{a_1}{z^2} + \cdots\right),$$

we find

$$\frac{1}{\zeta} = \frac{1}{z} - \frac{a_1}{z^3} + \cdots ; \quad \frac{1}{\zeta^2} = \frac{1}{z^2} - \frac{2a_1}{z^4} + \cdots ,$$

whence

$$\frac{df}{d\zeta} = - Ve^{i\alpha} - \frac{i\Gamma}{2\pi z} + \left(Va^2 e^{-i\alpha} - \frac{i\Gamma\zeta_0}{2\pi} \right) \frac{1}{z^2} + \cdots .$$

Multipliying this by

$$\frac{d\zeta}{dz} = 1 - \frac{a_1}{z^2} + \cdots ,$$

we find

$$w = - Ve^{i\alpha} - \frac{i\Gamma}{2\pi z} + \left(Va^2 e^{-i\alpha} - \frac{i\Gamma\zeta_0}{2\pi} + a_1 Ve^{i\alpha} \right) \frac{1}{z^2} + \cdots ,$$

$$w^2 = V^2 e^{2i\alpha} + \frac{iV\Gamma e^{i\alpha}}{\pi z} - \left(\frac{\Gamma^2}{4\pi^2} + 2V^2 a^2 - \frac{i\Gamma\zeta_0 Ve^{i\alpha}}{\pi} + 2a_1 V^2 e^{2i\alpha} \right) \frac{1}{z^2} + \cdots .$$

From this expression it follows, by (34), that

$$\begin{aligned} \overline{F} &\equiv F_x - iF_y = - i\rho V\Gamma e^{i\alpha} , \\ F &\equiv F_x + iF_y = i\rho V\Gamma e^{-i\alpha} . \end{aligned} \tag{36}$$

In other words *the force per unit length exerted by the fluid on the cylindrical obstacle has the magnitude* $\pm\rho V\Gamma$ *and is perpendicular to the direction of flight*, the sense of rotation from the direction of flight to the direction of the force being that of the circulation Γ. See Fig. 2. This force per unit length is called the *lift* per unit length and is denoted by L, so that we have the Kutta-Joukowski formula

$$L = \rho V\Gamma . \tag{37}$$

If the potential function f were assumed uniform, Γ would be zero and there would be no force experienced by the obstacle; a result known as d'Alembert's paradox. The component of the force in the direction opposite to the direction of flight is called the *drag* and the drag per unit length is de-

noted by D. We see, then, that the present theory gives $D=0$, which is not in accord with experience. It will appear in the next section that when account is taken of the fact that the length of the obstacle is not infinite this discrepancy can be removed.

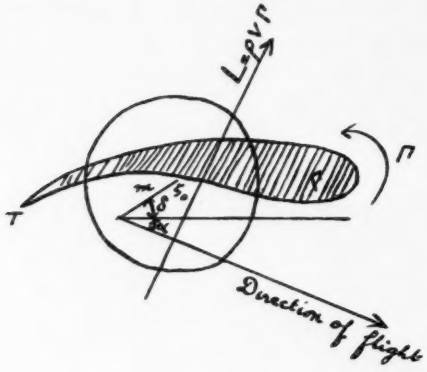

FIG. 2

If we introduce the notation $\zeta_0 = me^{i\delta}$, $a_1 = -b^2e^{2i\gamma}$, where m, δ, b and γ are all real, it follows that

$$M = \rho V\Gamma m \cos(\alpha+\delta) + 2\pi\rho b^2V^2 \sin 2(\alpha+\gamma),$$

and hence the turning moment about the center $\zeta_0=0$ of the circle in the auxiliary ζ plane, the two planes being supposed superposed, is

$$M_0 = 2\pi\rho b^2V^2 \sin 2(\alpha+\gamma). \tag{38}$$

11. *The Circular Arc and Straight Line Profile.* The relationship $((\zeta-l)/(\zeta+l))^2 = A(z-c/2)/(z+c/2)$, where A is any constant, maps a doubly covered circular arc through the points $\pm c/2$ in the z plane on a circle through the points $\pm l$ in the ζ plane. In order that the mapping may be of the type (32) we must have $A=1$, $l=c/4$ and the mapping is

$$z = \zeta + \frac{c^2}{16\zeta}, \quad \zeta = z - \frac{c^2}{16z} + \cdots, \tag{39}$$

it being necessary to take that one of the two determinations of ζ which tends to ∞ with z. If we denote by 2ϵ the angle of the circular arc, we have

$$(40)\quad \zeta_0 = il \operatorname{ctn} \epsilon = \frac{ic}{4} \operatorname{ctn} \epsilon, \qquad a = l \csc \epsilon = \frac{c}{4} \csc \epsilon,$$

$$\beta = \frac{\pi}{2} - \epsilon, \qquad b^2 = \frac{c^2}{16}, \qquad \gamma = 0,$$

so that, by (31)

$$(41)\quad \Gamma = \pi c V \csc \epsilon \cos (\alpha - \epsilon),$$

$$(42)\quad L = \pi \rho V^2 c \csc \epsilon \cos (\alpha - \epsilon), \qquad M_0 = (\pi \rho c^2 V^2 \sin 2\alpha)/8.$$

The distance h of the lifting force $L = \rho V \Gamma$ from the center of the auxiliary circle is

$$h = M_0/L = c \sin 2\alpha \sin \epsilon / [8 \cos(\alpha - \epsilon)].$$

The particular case where $\epsilon = \pi/2$ is very important. Here we are dealing with a plane wing flying at an *angle of attack* α and we find

$$(43)\quad \Gamma = \pi c V \sin \alpha, \qquad L = \pi c \rho V^2 \sin \alpha, \qquad h = c \cos \alpha/4.$$

The number c is the length of the chord of the wing and the expression for h indicates that the point where the line of action of L meets the chord, that is, the center of pressure, does not vary with α but remains constantly one-quarter of the length of the chord from the leading edge; for ζ_0 is now the middle point of the chord. The number $c_L = L/(\frac{1}{2}\rho V^2 c)$ is called the *lift coefficient* c_L and we have the result that for a plane wing

$$(44)\quad c_L = 2\pi \sin \alpha.$$

For small angles of attack we may put

$$(45)\quad c_L = 2\pi\alpha.$$

so that there is a linear relation connecting c_L and α. This prediction is verified by experience for angles of attack up to about 15°; but the factor of proportionality 2π is found to be about 40% too large. The reason for this discrepancy is that the wing is not infinite in length and a corrected expression will be given in the next section.

12. *Joukowsky Profiles and their Extensions.* The relationship between z and ζ which is expressed by (39) maps the exterior of a circle through the points $\pm c/4$ in the ζ plane upon the exterior of a doubly covered circular arc, with ends at the points $\pm c/2$, in the z plane. A profile suitable for airfoils is obtained if the same mapping relationship is applied to a circle through the point $-c/4$ but *not* through the point $+c/4$ in the ζ plane. If the circle does not differ very much from the circle through the points $\pm c/4$ which maps into the doubly covered circular arc, the contour will have a rounded leading edge and a sharp trailing edge. A linear fractional substitution on a complex variable being resolvable into a translation, rotation, reflexion in a line and an inversion may be called a generalized inversion and it is evident that the mapping (39) may be written as

$$z' = \zeta'^2, \quad z' = \frac{z - \frac{c}{2}}{z + \frac{c}{2}}, \quad \zeta' = \frac{\zeta - \frac{c}{4}}{\zeta + \frac{c}{4}}. \tag{46}$$

Now a generalized inversion sends circles into circles (straight lines being regarded as circles through the point at infinity). The generalized inversion in the ζ plane, written down in (46), sends any circle through $-c/4$ in this plane into a straight line and this is sent by the square-mapping $z' = \zeta'^2$ into a parabola in the z' plane. The profile obtained is then the generalized inverse of a parabola. Such profiles are known as Joukowsky profiles. Generalizations are obtained by writing, instead of (46),

$$(47) \qquad z' = \zeta'^n, \quad z' = \frac{z - \frac{c}{n}}{z + \frac{c}{n}}, \quad \zeta' = \frac{\zeta - \frac{c}{2n}}{\zeta + \frac{c}{2n}},$$

which is of the form (32) near $z = \infty$. This sends the exterior of a circle through the points $\pm c/(2n)$ in the ζ plane into the exterior of a region bounded by two circular arcs through the points $\pm c/n$ in the z plane, the angle between the two arcs being $(2-n)\pi$. By taking a circle in the ζ plane through $-c/(2n)$ but not exactly through $+c/(2n)$ we obtain a profile with a rounded leading edge and a sharp trailing edge; it has the crescent formed by the two circular arcs as a "skeleton" just as the Joukowsky profile has the doubly covered circular arc as skeleton. The essential difference is that the trailing edge is now a double point of the contour and not a cusp as in the case of a Joukowsky profile. The new profiles are obtained by applying a generalized inversion to the curves obtained by applying the nth power mapping $z' = \zeta'^n$ to a straight line (passing nearly through the origin) in the ζ' plane.

C. The Induced Drag of a Finite Airfoil

13. *The Formulas for the Lift and Drag.* It has been seen in the previous section that when the length of the airfoil is assumed infinite it experiences, owing to the circulation around it, a force at right angles to the direction of flight; in other words there is a lift but no drag. In order to give an explanation of this discrepancy between theory and experiment Prandtl advanced the idea that since an actual airfoil is not infinite in length the circulation around it may vary from point to point along its length. Under this hypothesis the flow loses at once its two-dimensional character but it is assumed that the results obtained for the infinite wing may be used as first approximations; the fact that the circulation is not constant along the span of the airfoil introduces a correction to this approximation. To get an

estimate of the nature and amount of this correction we shall consider the case of a plane wing whose chord c is small in comparison with its length or *span* s and we shall suppose the airfoil sufficiently represented by its trailing edge (it is then called a lifting line). If we introduce a coordinate system with its origin at the middle of the lifting line (see Fig. 3), we may draw the x axis in the direction of flight, the z axis vertically downwards and the y axis along the lifting line. The circulation around the wing in any plane perpendicular to it is assumed to be a function $\Gamma(y)$

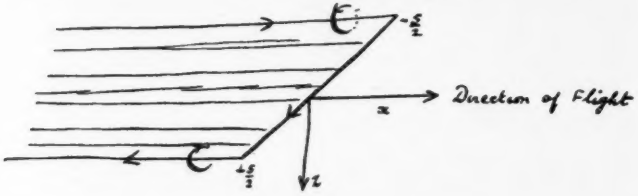

FIG. 3

of y. If then we consider the ring-shaped area bounded by two closed curves enlacing the wing, one in the plane y and the other in the plane $y+\Delta y$, there will be a flux of vorticity through this area, outwards from the wing, of amount $-(d\Gamma/dy)\Delta y$. (See (8).) We assume that this flux is concentrated at the trailing edge and is in the direction of the negative x axis, that is, is in the direction opposite to the direction of flight. Hence there will be a sheet of vortices trailing off from the trailing edge of the wing such that the strength of the vortices leaving an element Δy is $-(d\Gamma/dy)\Delta y$. Now if we have a single rectilinear vortex filament of strength μ in an indefinitely extended fluid which is at rest at infinity, the velocity distribution must be symmetrical around it and hence at a distance r from it the velocity must be of magnitude $\mu/(2\pi r)$. We *assume* that for a *semi*-infinite filament extending, as in the present case, along the negative x axis, the magnitude of the velocity at a point along the y axis distant r from the filament will be $\mu/(4\pi r)$. The direction

of the velocity will be that of the positive or negative z axis. This argument leads to the conclusion that the presence of the semi-infinite strip of trailing vortices induces at any point y of the lifting line a velocity $w(y)$ in the direction of the z axis, that is downwards, of amount

$$w(y) = \frac{1}{4\pi}\int_{-s/2}^{s/2} \frac{(d\Gamma/dy)_{y=y'}}{y - y'}dy'. \tag{48}$$

This assumption of an induced *downwash* is the cardinal one of Prandtl's theory for the finite wing. For an infinite plane wing flying at a small angle of attack α with uniform velocity V an easy evaluation of the indeterminate form df/dz shows that the relative velocity of the wind at the trailing edge has the magnitude $V \cos \alpha$ and the direction of the negative chord of the wing and for small angles of attack α this can be regarded as of magnitude V in the direction opposite to the direction of flight. The net effect of compounding with this relative velocity V the downwash w is to rotate the relative velocity through the angle $\tan^{-1}(w/V)$ and since w is merely a small corrective term this can be equated to w/V. This rotation of the relative wind, due to the sheet of trailing vortices, may be regarded as equivalent to a rotation of the direction of flight through the same angle, that is to a diminution of the angle of attack (which is the angle from the direction of flight to the chord of the wing). The diminished angle α_e is called the *effective* angle of attack (as opposed to the *geometric* angle of attack α_g) and we have the equation

$$\alpha_e = \alpha_g - \frac{w}{V}. \tag{49}$$

On making the assumption that we can apply the results developed for the infinite wing, once we have replaced the geometric angle of attack by the effective angle of attack, we find that the lift per unit length L' at any point y on the span of the wing is given by

$$2\pi c \cdot \frac{1}{2}\rho V^2 \cdot \left(\alpha_g - \frac{w}{V}\right),$$

and on using $L' = \rho V \Gamma$ and equation (48), we have the equation

$$\Gamma(y) = \pi c V \alpha_g - \frac{c}{4} \int_{-s/2}^{s/2} \frac{(d\Gamma/dy)_{y=y'}}{y - y'} dy' \tag{50}$$

for the unknown function $\Gamma(y)$. For a given airfoil both c and α_g will be given functions of y; if α_g is not constant the airfoil is said to be twisted. It is apparent from (49) that even when α_g is constant α_e may vary owing to the possibility of a variable downwash $w(y)$. This variability of α_e is known as aerodynamic twisting of the airfoil.

Since the lift experienced by an infinite wing is perpendicular to the direction of flight and since the effect of the downwash is to decrease, in effect, the angle of attack by w/V, the direction of the force at any point of the wing will be rotated through an angle w/V from the direction of lift. It is this rotation of the direction of the force which introduces a component in the direction opposite to that of flight. Once $\Gamma(y)$ is determined from (50), the total lift L and drag D experienced by the wing are given by the expressions

$$L = \rho V \int_{-s/2}^{s/2} \Gamma dy, \qquad D = \rho \int_{-s/2}^{s/2} w \Gamma dy, \tag{51}$$

(where we have identified the sine of w/V with w/V). The drag D given by (51) is known as *induced* drag, it being a matter of experience that it is too small by a certain amount depending on the shape of the airfoil and known as the profile drag.

14. *The Solution of the Integral Equation and the Determination of the Airfoil of Minimum Induced Drag having a Given Lift.* In order to obtain a numerical solution of (50) we make the substitution $y = -(s/2)\cos\theta$, so that θ runs from 0 to π along the span of the wing and imagine $\Gamma(y) = F(\theta)$ developed

in a Fourier sine series (in order that there may be no discontinuity in $\Gamma(y)$ at the wing tips, $y=\pm s/2$, we must have $\Gamma(\pm y/2)=0$). Writing this series as $A_1 \sin\theta+A_2 \sin 2\theta + \cdots$, we see from the fact that $\Gamma(y)$ is symmetric about the middle of the wing that $A_2=A_4=A_6=\cdots=0$. In actual calculations it will be sufficient to consider the four terms $A_1 \sin\theta+A_3 \sin 3\theta+A_5 \sin 5\theta+A_7 \sin 7\theta$ in the development of $\Gamma(y)$. A simple substitution in (50) and a knowledge of the fact that

$$\int_0^\pi \frac{\cos n\phi d\phi}{\cos\phi-\cos\theta} = \pi \frac{\sin n\theta}{\sin\theta}$$

gives the equation

$$(52) \qquad \sum_1^\infty A_n \sin n\theta \left\{ \sin\theta + n\frac{\pi c}{2s} \right\} = \pi c V \alpha_g \sin\theta$$

for the determination of the coefficients A_n. On substituting the values $\theta=\pi/8, \theta=\pi/4, \theta=3\pi/8, \theta=\pi/2$, in this and dropping the terms after A_7 we have four linear equations for the determination of the four coefficients A_1, A_3, A_5, and A_7.

The coefficients A_n being supposed known, the expressions for the lift L and the drag D in terms of them follow readily from the formulas (51). We find

$$(53) \qquad L = \rho V s A_1/4, \quad D = \frac{\pi\rho}{8} \sum_1^\infty n A_n^2,$$

so that the lift depends only on the first term in the Fourier development of $\Gamma(y)\equiv F(\theta)$. The downwash w is given by

$$(54) \qquad w(y) = \frac{1}{2s} \sum_1^\infty \frac{n A_n \sin n\theta}{\sin\theta},$$

and the drag D by

$$(55) \qquad D = \frac{\pi\rho}{8} \sum_1^\infty n A_n^2.$$

It follows that D is a minimum, for a given value of L, when

the remaining coefficients $A_3, A_5, \cdots$ after A_1 in the Fourier development of $F(\theta)$, are zero. In this case the downwash has the constant value $A_1/(2s)$ and there is no aerodynamic twisting of the wing. Since

$$\Gamma(y) = A_1 \sin\theta = A_1\left(1 - \frac{4y^2}{s^2}\right)^{1/2},$$

the graph of $\Gamma(y)$ against y is an ellipse and the wing is said to be elliptically loaded. Since for an infinite plane wing the lift per unit length is, for a given angle of attack, proportional to the chord c (see (43)), we may surmise that an elliptically loaded wing would be approximated to by one whose profile is approximately elliptical. A criterion of good design of an airfoil is that, for a given L, D should be as small as possible and so for a well designed airfoil we may assume the loading approximately elliptical.

15. *Comparison with Experiment.* For a finite wing having a constant chord c the ratio of the span s to the chord c is known as the aspect ratio. For a plane wing whose chord is not constant we first define the mean chord c as the quotient of the area S of the wing by its span s; then the aspect ratio A is the ratio of the span of the wing to its mean chord. Hence $A = s^2/S$. The lift coefficient c_L of the wing is defined by the equation

$$c_L = L/(\tfrac{1}{2}\rho V^2 S), \tag{56}$$

and there is a similar equation defining the drag coefficient. For an elliptically loaded wing we have, by (53) and (56),

$$c_L = \frac{\pi A}{2Vs} A_1, \tag{57}$$

while the downwash w has the constant value

$$w = \frac{A_1}{2s} = \frac{V}{\pi A} c_L. \tag{58}$$

Hence the diminution in the angle of attack, induced by the

trailing vortices, is $c_L/(\pi A)$. The total lift L of the wing is

$$2\pi\alpha_e \cdot \tfrac{1}{2}\rho V^2 \cdot \int_{-s/2}^{s/2} c\,dy = 2\pi\alpha_e \cdot \tfrac{1}{2}\rho V^2 \cdot S,$$

so that $c_L = 2\pi\alpha_e$ and we have

$$\alpha_g = \alpha_e + \frac{c_L}{\pi A} = \alpha_e\left(1 + \frac{2}{A}\right). \tag{59}$$

The aspect ratio of a wing is usually about 6, so that the lift formula for the infinite wing (using α_g) might be expected to give a result which is too large in the ratio 4:3. This is confirmed by experience.

The drag coefficient c_D has, for an elliptically loaded wing, the value $\pi\rho A_1^2/8$ (by (55)) so that, by (57),

$$c_D = c_L^2/(\pi A). \tag{60}$$

Hence a curve plotting D against L should be parabolic and this is verified experimentally; the actual curve obtained empirically indicates a relationship of the type

$$c_D = c_{D_0} + c_L^2/(\pi A), \tag{61}$$

and c_{D_0} is called the coefficient of profile drag. The formulas (59) and (61) are of importance because they enable us to calculate the effect of different aspect ratios; thus for two airfoils having the same lift coefficient

$$\alpha_g' - \alpha_g = \frac{c_L}{\pi}\left(\frac{1}{A'} - \frac{1}{A}\right), \quad c_D' - c_D = \frac{c_L^2}{\pi}\left(\frac{1}{A'} - \frac{1}{A}\right). \tag{62}$$

D. Concluding Remarks

In the present address, I have tried to present in an elementary way some of those fundamental principles of modern hydrodynamical theory, which have proved useful in aeronautics, in the hope of interesting some of our hearers and readers in these questions. It would be easy to point out many unsolved problems and indeed a casual reading of the previous sections will show that even in the most elementary

problems assumptions have continually to be made the legitimacy of which is at least not proved. We append a bibliography which has been deliberately restricted to those works which we would recommend to a student wishing to orient himself in the subject and to reach research problems as soon as possible.

TEXTBOOKS

I. H. Lamb, *Hydrodynamics*. 5th ed. Cambridge University Press. The standard text on hydrodynamics.

II. H. Glauert, *The Elements of Aerofoil and Airscrew Theory*. Cambridge University Press. A fine and elementary introduction to the modern problems and methods.

III. P. Appell, *Traité de Mécanique rationnelle*, vol. 3. Gauthier Villars.

IV. L. Bairstow, *Applied Aerodynamics*. Longmans-Green.

V. R. von Mises, *Fluglehre*. Julius Springer.

These two texts are less mathematical and more physical than the other texts and are particularly valuable as complementing them.

VI. Fuchs-Hopf, *Aerodynamik*. R. C. Schmidt. This is volume 2 of a *Handbuch der Flugzeugkund*, and is very complete.

VII. A. Betz, *Tragflügel und Hydraulische Maschinen*. Handbuch der Physik, vol. 7, pp. 215–288.

VIII. Th. v. Karman and T. Levi-Civita, *Hydro- und Aerodynamik*. J. Springer.

Many papers of interest will be found in the volumes of the Zeitschrift für Angewandte Mathematik and in the publications of the National Advisory Commitee for Aeronautics. Of these latter the following may be especially recommended:

IX. N.A.C.A. Report No. 116 (Prandtl). Government Printing Office, Washington, D.C.

X. N.A.C.A. Report No. 213 (Ames). Government Printing Office, Washington, D.C.

THE JOHNS HOPKINS UNIVERSITY

THE EQUATION OF THE V_{n-1}^{n-1} IN S_n*

BY C. A. RUPP

Segre, dealing with synthetic and enumerative geometry, has twice mentioned† that the manifold formed by all the lines of S_n which meet n given S_{n-2}'s is a V_{n-1}^{n-1}. In a recent article, Wong‡ mentions that the equation of the general V_{n-1}^{n-1} is unknown. It is the purpose of this paper to exhibit the equation.

The method is the following: through a general point of one of a set of n given S_{n-2}'s is passed a line which is required to meet each of the remaining $(n-1)$ S_{n-2}'s. The eliminant of the system of equations thus set up is the desired equation, as will shortly appear.

Let the equations of the n given S_{n-2}'s be

$$x_i = 0 = \sum_{j=0}^{j=n} a_{ij}x_j, \qquad (i = 1,2, \cdots, n), \qquad (a_{ii} = 0).$$

Berzolari§ has shown that the above display does not particularize the set of S_{n-2}'s, but is a mere question of a suitable choice of the reference system. We shall need the Grassmann-Plücker coordinates of the S_{n-2}'s, that is, the two-rowed determinants from the matrices of the coefficients in their equations. In each set of coordinates we here find $n(n-1)/2$ elements are zero; the remaining n are some of the numbers a_{ij}, prefaced with a proper sign.

Let y be the coordinates of a point on one of the S_{n-2}'s, say the first one. We then have that

$$(A) \qquad y_1 = 0, \qquad \sum_{j=0}^{j=n} a_{1j}y_j = 0.$$

* Presented to the Society, December 27, 1928.

† Segre, *Mehrdimensionale Räume*, Encyklopädie, vol. III C7, pp. 815 and 832.

‡ Wong, B. C., this Bulletin, vol. 34 (1928), pp. 553–554.

§ Berzolari, Rendiconti del Circolo Matematico di Palermo, vol. 29 (1905), p. 229.

Let x be the coordinates of a point on a line through y, and distinct from y. The coordinates of this line may be denoted by

$$p_{ij} = x_i y_j - x_j y_i, \qquad (i,j = 0,1,\cdots, n),\ (i \neq j).$$

The condition that this line meet an S_{n-2} is that the summation of the product of corresponding coordinates vanish. When we stipulate that the line shall meet all the S_{n-2}'s of the set, we adjoin to equations (A) the $n-1$ equations (B), where

$$\text{(B)} \qquad \sum_{j=0}^{j=n} a_{ij} p_{ij} = 0, \qquad (i = 2,3,\cdots, n).$$

When we eliminate y from the two sets of equations, the resulting equation in x represents the equation of the V_{n-1}^{n-1}, for the line from x to y has been made to meet each of the n S_{n-2}'s and is hence a generator of the V_{n-1}^{n-1}. One may readily verify that the final form of the eliminant does not depend upon the particular member of the set of S_{n-2}'s whereon one chooses to place the point y.

If the interested reader will consider the following S_2's in S_5,

$$x_0 = x_1 = 0, \quad x_2 = x_3 = 0, \quad x_4 = x_5 = 0, \quad x_0 = x_2 = x_4, \quad x_1 = x_3 = x_5,$$

locate the point y on one of the S_2's, compute the coordinates of the line from x to y as well as the coordinates of the remaining four S_2's, write down the four incidence relations that cause the line to meet the four S_2's, eliminate y from the set, and reduce, he will have the pleasure of finding the equation of a V_4^4 in S_5; the result is

$$0 = x_0x_1x_2x_5 + x_0x_3x_4x_5 + x_1x_2x_3x_4 - x_0x_1x_3x_4 - x_0x_2x_3x_5 - x_1x_2x_4x_5.$$

THE PENNSYLVANIA STATE COLLEGE

A PARTIAL ISOMORPHISM BETWEEN THE FUNCTIONS OF LUCAS AND WEIERSTRASS*

BY E. T. BELL

1. *Introduction.* In a former paper† it was pointed out that certain identities in the theory of multiplication, real or complex, of elliptic functions, are of precisely the same form as others between Lucas' U_n, V_n. The latter were defined only for integer values of n by Lucas as follows,

$$U_n \equiv \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n \equiv \alpha^n + \beta^n,$$

where α, β are real or complex numbers such that $\alpha+\beta=p$, $\alpha\beta=q$, where p, q are rational numbers. Hence U_n, V_n are rational numbers and are solutions of

$$W_{n+2} = pW_{n+1} - qW_n,$$

with the initial conditions

$$(U_0, U_1) = (0,1), \ (V_0, V_1) = (2, p).$$

It follows, as shown by Lucas,‡ that U_n, V_n are simply expressible as circular functions of a real or imaginary argument, according as α/β is imaginary or real. Thus every formula in elliptic functions will contain as a degenerate case one concerning the U_n, V_n functions, but this is not the isomorphism sought.

We first generalize Lucas' definition in one respect and specialize it in another, replacing n by the complex variable z and restricting q to be 1. We shall write the functions thus defined as

$$(1) \qquad U(z) \equiv \frac{\alpha^z - \alpha^{-z}}{\alpha - \alpha^{-1}}, \quad V(z) \equiv \alpha^z + \alpha^{-z},$$

* Presented to the Society, December 31, 1928.

† This Bulletin, vol. 29 (1923), pp. 401–406.

‡ American Journal of Mathematics, vol. 1, (1878), p. 189.

where α, p are constants, real or complex, such that $\alpha+\alpha^{-1}=p$. The principal determinations of exponentials and logarithms are to be understood. From $U(z)$, $V(z)$ we shall construct certain functions exhibiting a partial isomorphism with the $\sigma(z)$, $\zeta(z)$, $\wp(z)$ of Weierstrass. A complete isomorphism is obviously impossible, $U(z)$, $V(z)$ being only singly periodic.

To make the sense in which we are using isomorphism precise, we recall a well known theorem for the sigma function.* Let $f'(z)$ denote the derivative of $f(z)$ with respect to z, and suppose that $f(z)$ satisfies the following conditions.

(A) $f(z)$ is a transcendental integral function of z.

(B) $f'(0) = 1$.

(C) $f(z+a)f(z-a)f(b+c)f(b-c)$
$+ f(z+b)f(z-b)f(c+a)f(c-a)$
$+ f(z+c)f(z-c)f(a+b)f(a-b) = 0,$

where z, a, b, c are independent complex variables. Then necessarily $f(z)\equiv\sigma(z)$. If now $f(z)$ satisfies (A) and (C), but not (B), we shall call $f(z)$ a *partial isomorph* of $\sigma(z)$. As shown by Kronecker,† the "three-term relation" (C) is formally equivalent to Jacobi's formula for the multiplication of four elliptic theta functions. Hence, if preferred, Jacobi's relation may replace (C) in what follows. The notation sh x, ch x is used for the hyperbolic sine and cosine of x.

2. *The Functions* λ, μ, ν. With α, p as in (1), define the constants k, g by

$$k \equiv \log\alpha, \qquad 2g \equiv k\,\mathrm{hs}\,k, \tag{2}$$

where $\mathrm{hs}\,x\equiv 1/\mathrm{sh}\,x$. Then $p=2\,\mathrm{ch}\,k$, and

$$W(z+2) - 2\,\mathrm{ch}\,kW(z+1) + W(z) = 0$$

has the solutions $W(z)\equiv U(z)$, $V(z)$.

* Halphen, *Traité des Fonctions Elliptiques*, vol. 1, p. 187.

† Journal für Mathematik, vol. 102, p. 260.

Write ch $x/\text{sh}\, x \equiv \text{cs}\, x$, and define the functions λ, μ, ν by

$$\lambda(z) \equiv \frac{1}{2g}U(z) = \frac{1}{k}\,\text{sh}\, kz, \tag{3}$$

$$\mu(z) \equiv \frac{g^2V^2(z)}{U^2(z)} - k^2 = k^2\,\text{hs}\, kz, \tag{4}$$

$$\nu(z) \equiv g\frac{V(z)}{U(z)} = k\,\text{cs}\, kz. \tag{5}$$

The hyperbolic forms can be verified by inspection from (2) and

$$U(z) = \frac{2g}{k}\,\text{sh}\, kz, \quad V(z) = 2\,\text{ch}\, kz. \tag{6}$$

3. *Correspondence between* λ, μ, ν *and* σ, $\wp$, ζ. We shall now show that $\lambda(z)$, $\mu(z)$, $\nu(z)$ correspond, up to a certain point, to $\sigma(z)$, $\wp(z)$, $\zeta(z)$ respectively. Note first that $\lambda'(0) = 1$, so that §1 (B) does not hold. The addition and subtraction theorems for $U(z)$ are

$$2U(z \pm w) = U(z)V(w) \pm V(z)U(w),$$

as is evident from (6), and therefore, if $K \neq 0$ and L are arbitrary constants, and if for the moment we write $W(z) \equiv KU(z)$, we have the identity

$$\left[\left\{\frac{V(z)}{2W(z)}\right\}^2 + L\right] - \left[\left\{\frac{V(w)}{2W(w)}\right\}^2 + L\right] = -\frac{W(z+w)W(z-w)}{W^2(z)W^2(w)}.$$

The correspondence will be effected by properly choosing K, L, as the last identity is of the same form as

$$\wp(z) - \wp(w) = -\frac{\sigma(z+w)\sigma(z-w)}{\sigma^2(z)\sigma^2(w)}, \tag{D}$$

from which the three-term relation for σ follows at once, in the usual manner, by means of the algebraic identity

$$(b-c)(d-a)+(c-a)(d-b)+(a-b)(d-c)\equiv 0.$$

The constants are determined by assuming $W(z)$ to be the correspondent of $\sigma(z)$, and recalling the definitions

$$\zeta(z)\equiv\sigma'(z)/\sigma(z),\qquad \wp(z)=-\zeta'(z). \tag{7}$$

A simple calculation gives the functions $\lambda(z)$, $\mu(z)$, $\nu(z)$ as defined in §2, (3)–(5). Hence, or directly from (3)–(5), we see that

(8) $\lambda(z)$ *satisfies* §1 (A), (C), *but not* (B), *so that* $\lambda(z)$ *is a partial isomorph of* $\sigma(z)$;

$$\nu(z)=\lambda'(z)/\lambda(z),\qquad \mu(z)=-\nu'(z), \tag{9}$$

corresponding to (7);

$$\lambda(-z)=-\lambda(z),\quad \nu(-z)=-\nu(z),\quad \mu(-z)=\mu(z), \tag{10}$$

corresponding to the like for $\sigma(z)$, $\zeta(z)$, $\wp(z)$ *respectively*;

$$\mu'^2(z)=4\mu^3(z)+4k^2\mu^2(z)\ ; \tag{11}$$

and hence, if $\theta(z)\equiv\mu(z)+k^2/3$,

$$\theta'^2(z)=4\theta^3(z)-k_1\theta(z)-k_2, \tag{12}$$

corresponding to the Weierstrass normal form with the invariants $k_1=4k^4/3$, $k_2=(-2k^2/3)^3$.

The consequences of this correspondence can be developed indefinitely, without further computations. For example, since the correspondent of (D) is

$$\mu(z)-\mu(w)=-\frac{\lambda(z+w)\lambda(z-w)}{\lambda^2(z)\lambda^2(w)}, \tag{13}$$

we infer the following correspondent for the addition theorem for the $\wp$ function in one of its usual forms,

$$\mu(z\pm w)=\mu(z)-\frac{1}{2}\,\frac{d}{dz}\left[\frac{\mu'(z)\mp\mu'(w)}{\mu(z)-\mu(w)}\right],$$

and the like with θ in place of μ. Again, since the limit of $\lambda(t)/t$ as t approaches 0 is 1, we divide (13) throughout by

$z-w$, take the limit as w approaches z, and find $\mu'(z) = -\lambda(2z)/\lambda^4(z)$, the correspondent of $\wp'(z) = -\sigma(2z)/\sigma^4(z)$.

If we wish to interpret this partial isomorphism in terms of Lucas' U_n, V_n, where n is an integer, we may so do in the special case (cf. §1) in which $q=1$. The derivatives $\lambda'(z)$, $\mu'(z)$, $\cdots$ are then replaced by their equivalents in terms of $U(z)$, $V(z)$ with z finally replaced by n.

CALIFORNIA INSTITUTE OF TECHNOLOGY

AN INEQUALITY FOR DEFINITE HERMITIAN DETERMINANTS*

BY E. F. BECKENBACH

The proof of M. Ragnar Frisch's theorem,† *The absolute value of a symmetric, definite determinant of real elements is at most equal to the product of the absolute values of the elements of the principal diagonal,* may be generalized to establish the following theorem of which the above is clearly a special case: *The absolute value of a definite Hermitian determinant is at most equal to the product of the absolute values of the elements of the principal diagonal.*

An Hermitian determinant

$$H \equiv \begin{vmatrix} h_{11} \cdots h_{1n} \\ \cdots\cdots\cdots \\ h_{n1} \cdots h_{nn} \end{vmatrix} \tag{1}$$

is a determinant such that

$$h_{rs} = \bar{h}_{sr}, \qquad (r,s = 1,2, \cdots, n),$$

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* Presented to the Society, February 23, 1929.

† *Sur le théorème des déterminants de M. Hadamard,* Comptes Rendus, vol. 185 (1927), p. 124. This is not a new theorem; see Bachmann, *Die Arithmetik der Quadratischen Formen,* 1923, pp. 250–251. It is more the method of proof than the result that makes Frisch's paper of interest. Though Bachmann does not give the generalized proof of the present paper, his proof holds equally for it.

where $\bar{h}_{sr}$ designates the conjugate imaginary of h_{sr}. A definite Hermitian determinant is an Hermitian determinant such that the corresponding Hermitian form

$$h \equiv \sum_{r,s=1}^{n} h_{rs} x_r \bar{x}_s, \qquad (h_{rs} = \bar{h}_{sr}), \tag{2}$$

is definite. The reader may readily verify the fact that an Hermitian form can never take on a complex value. An Hermitian form which is positive (negative) or zero for all values of the variables is said to be positive (negative) definite; in either case, the form is said to be definite.

We can suppose the definite form (2) corresponding to the definite determinant (1) to be positive definite, for otherwise we could consider the form

$$-h \equiv \sum_{r,s=1}^{n} (-h_{rs}) x_r \bar{x}_s, \qquad (h_{rs} = \bar{h}_{sr}),$$

which would be positive definite, and whose corresponding determinant would be equal in absolute value to H. And we can suppose the form to be non-singular, for if the determinant were equal to zero the theorem would be trivial.

The determinant

$$F(\lambda) \equiv \begin{vmatrix} h_{11} - \lambda & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} - \lambda & \cdots & h_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} - \lambda \end{vmatrix}$$

may be expanded into a polynomial in λ of degree n, where the constant term, H, is unequal to zero. Hence we can construct*

$$F(\lambda) \equiv \frac{[P(\lambda)]^2 - \lambda}{Q(\lambda)}, \tag{3}$$

where $P(\lambda)$ and $Q(\lambda)$ are polynomials in λ and $P(\lambda)$ is of

* See Bôcher, *Introduction to Higher Algebra*, pp. 297–299.

degree less than n. As is well known,* the roots of the characteristic equation,

$$F(\lambda) = 0,$$

are all real and greater than zero. Note, then, in the above construction, that the coefficients in $P(\lambda)$ are all real.

Let

$$M \equiv \begin{pmatrix} h_{11} \cdots h_{1n} \\ \cdots\cdots\cdots \\ h_{n1} \cdots h_{nn} \end{pmatrix}$$

be the matrix corresponding to H. By a fundamental theorem,†

$$F(M) \equiv 0.$$

Hence, (3) gives

$$[P(M)]^2 \equiv M. \tag{4}$$

It is easy to show that $P(M)$, being a polynomial with real coefficients in an Hermitian matrix, is itself an Hermitian matrix of the same order. Now let

$$P(M) \equiv \begin{pmatrix} b_{11} \cdots b_{1n} \\ \cdots\cdots\cdots \\ b_{n1} \cdots b_{nn} \end{pmatrix}, \qquad (b_{rs} = \bar{b}_{sr}).$$

Then from (4),

$$h_{ii} = \sum_{j=1}^{n} b_{ij}\bar{b}_{ij}, \qquad (i = 1, 2, \cdots, n).$$

Whether the form is positive or negative definite, we still have

$$\text{mod } h_{ii} = \sum_{j=1}^{n} b_{ij}\bar{b}_{ij}, \qquad (i = 1, 2, \cdots, n), \tag{5}$$

where by mod h_{ii} we mean the modulus, or absolute value, of h_{ii}.

* See Kowalewski, *Einführung in die Determinantentheorie*, Leipzig, 1909, pp. 280–284.

† See Bôcher, loc. cit., p. 296.

I am indebted to Professor H. E. Bray for the following proof of Hadamard's Theorem. Given the matrix of real or complex elements

$$\alpha \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{pmatrix}, \qquad \alpha_i \equiv (\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in}),$$

consider the matrix

$$\beta \equiv \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_n \end{pmatrix},$$

where

$$\beta_1 = \alpha_1,$$
$$\cdots\cdots\cdots$$
$$\beta_i = \sum_{j=1}^{i-1} k_j^i \beta_j + \alpha_i.$$

We note that

$$|\beta| = |\alpha|.$$

Choose the k_j^i so that, for given i and all $j<i$,

$$(\beta_i, \bar{\beta}_j) \equiv \beta_{i1}\bar{\beta}_{j1} + \beta_{i2}\bar{\beta}_{j2} + \cdots + \beta_{in}\bar{\beta}_{jn} = 0.$$

That is,

$$\left(\left[\sum_{l=1}^{i-1} k_l^i \beta_l + \alpha_i\right], \bar{\beta}_j\right) = k_j^i(\beta_j, \bar{\beta}_j) + (\alpha_i, \bar{\beta}_j) = 0,$$

or

$$k_j^i = -\frac{(\alpha_i, \bar{\beta}_j)}{(\beta_j, \bar{\beta}_j)}.$$

Hence

$$\sum_{j=1}^{i-1} \bar{k}_j^i (\alpha_i, \bar{\beta}_j) = -\sum_{j=1}^{i-1} k_j^i \bar{k}_j^i (\beta_j, \bar{\beta}_j),$$

which is real and negative. Now

$$0 \leqq (\beta_i, \bar{\beta}_i) = \left(\left[\sum_{j=1}^{i-1} k_j^i \beta_j + \alpha_i\right], \left[\sum_{j=1}^{i-1} \bar{k}_j^i \bar{\beta}_j + \bar{\alpha}_i\right]\right),$$

whence

$$0 \leqq (\beta_i \bar{\beta}_i) = \sum_{j=1}^{i-1} k_j^i \bar{k}_j^i (\beta_j, \bar{\beta}_j) + \sum_{j=1}^{i-1} \bar{k}_j^i (\alpha_i, \bar{\beta}_j) + \sum_{j=1}^{i-1} k_j^i (\bar{\alpha}_i, \beta_j) + (\alpha_i, \bar{\alpha}_i)$$

$$= (\alpha_i, \bar{\alpha}_i) - \sum_{j=1}^{i-1} k_j^i \bar{k}_j^i (\beta_j, \bar{\beta}_j) \leqq (\alpha_i, \bar{\alpha}_i).$$

But

$$|\alpha| \cdot |\bar{\alpha}| = |\beta| \cdot |\bar{\beta}| = \prod_{i=1}^{n} (\beta_i, \bar{\beta}_i).$$

Hence

$$|\alpha| \cdot |\bar{\alpha}| \leqq \prod_{i=1}^{n} (\alpha_i, \bar{\alpha}_i),$$

or

$$\operatorname{mod} |\alpha| \leqq \prod_{i=1}^{n} [(\alpha_i, \bar{\alpha}_i)]^{1/2}.$$

This inequality expresses Hadamard's Theorem: *The absolute value of any determinant is at most equal to the product of the norms (either horizontal or vertical) of the determinant.* The quantities

$$+ [(\alpha_i, \bar{\alpha}_i)]^{1/2}, \quad \text{or} \quad + \left(\sum_{j=1}^{n} \alpha_{ij} \bar{\alpha}_{ij} \right)^{1/2}, \quad (i = 1, 2, \cdots, n),$$

are called the horizontal norms of the determinant $|\alpha|$, and a similar definition holds for the vertical norms.

Applying this theorem to the determinant $|P(M)|$, we get

$$\operatorname{mod} |P(M)|^2 \leqq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} b_{ij} \bar{b}_{ij} \right). \tag{6}$$

Equations (4), (5), and (6) give

$$\operatorname{mod} H \leqq \prod_{i=1}^{n} (\operatorname{mod} h_{ii}). \qquad Q.E.D.$$

THE RICE INSTITUTE

A THEOREM CONCERNING SIMPLY TRANSITIVE PRIMITIVE GROUPS*

BY W. A. MANNING

The theorem here presented has evolved by easy stages from a paragraph in Jordan's *Memoir on primitive groups.*† In the discussion of a particular class of simply transitive primitive groups, he showed that the degree of a doubly transitive constituent of the subgroup leaving one letter fixed cannot be greater than the sum of the degrees of all the other transitive constituents.

THEOREM. *Let H be the subgroup that fixes one letter of a simply transitive primitive group. If one of the constituents of H is a doubly transitive group of degree m, there is in H a transitive constituent whose degree is greater than m and divides $m(m-1)$.*

Let G, of degree n and of order nh, be the given simply transitive primitive group. Let H, the subgroup of G that leaves the letter x fixed, be denoted by $G(x)$.

Let it be assumed: (1) that $G(x)$ has exactly k similar‡ doubly transitive constituent groups: A on the letters $a_1, a_2, \cdots, a_m$; B on $b_1, b_2, \cdots, b_m$; $\cdots$; K on $k_1, k_2, \cdots, k_m$; (2) that $G(a_1)$ has $k-1$ doubly transitive constituents: B_1 on $b_1, a_2, \cdots, a_m$; C_1 on $c_1, b_2, \cdots, b_m$; $\cdots$; K_1 on $k_1, j_2, \cdots, j_m$; (3) that n is greater than $km+1$. These assumptions, when $k=1$, reduce to the hypothesis of our theorem. We wish to prove by induction that there is a transitive

* Presented to the Society, April 6, 1928.

† C. Jordan, Bulletin de la Société Mathématique de France, vol. 1 (1873), p. 198, §64.

Manning, American Journal of Mathematics, vol. 39 (1917), p. 298; Transactions of this Society, vol. 20 (1919), p. 66; *Primitive Groups*, 1921, p. 83; Transactions of this Society, vol. 29 (1927), p. 821, §8.

‡ Manning, Transactions of this Society, vol. 29 (1927), p. 821, §8.

constituent of some degree μ in $G(x)$ such that μ divides $m(m-1)$ and is greater than m.

The letters of G are supposed to be so chosen that each substitution of $G(x)$ permutes the m subscripts of the letters of the similar groups $A, B, \cdots, K$ in exactly the same way.

The subgroup $G(x)(a_1)$ of $G(x)$ (that fixes a_1) is of order h/m. The subgroups $G(x)(a_1)$ and $G(a_1)(x)$ are identical, and therefore in $G(a_1)$ x is a letter of a transitive constituent of degree m. $G(a_1)(x)(=G(b_1)(x)=G(c_1)(x)=\cdots=G(k_1)(x))$, because of assumption (1) has k transitive constituents of degree $m-1$ each on the letters $a_2, a_3, \cdots, a_m$; $b_2, b_3, \cdots, b_m$; $\cdots$; $k_2, k_3, \cdots, k_m$, respectively. Since G is primitive, $\{G(x), G(a_1)\}=G$, and therefore $G(a_1)$ cannot have a transitive constituent of degree $m-1$ on the letters $k_2, k_3, \cdots, k_m$, nor a transitive constituent of degree m on the letters $x, k_2, \cdots, k_m$. In $G(a_1)$ the letters $k_2, k_3, \cdots, k_m$ all belong to the same transitive constituent of unknown degree $\mu(\geqq m)$. Then $G(a_1)(k_2)$ is of order h/μ and if x belongs to a transitive constituent of degree $\delta(\geqq 1)$ in $G(a_1)(k_2)$, $G(a_1)(k_2)(x)$ is of order $h/\mu\delta$. Then since $G(x)(a_1)(k_2)=G(x)(k_1)(k_2)$, $h/\mu\delta=h/[m(m-1)]$, and μ divides $m(m-1)$. Either our theorem is proved or $\mu=m$. If $\mu=m$, $G(a_1)$ has this transitive constituent (L_1) on the letters $l_1, k_2, \cdots, k_m$ (where $l_1\neq x$); L_1 is doubly transitive because it contains a transitive subgroup of degree $m-1$ on $k_2, k_3, \cdots, k_m$.

The substitution $(a_1a_i\cdots)\cdots$ of $G(x)$ transforms $G(a_1)$ into $G(a_i)$ and L_1 into L_i, a doubly transitive constituent of $G(a_i)$ on the m letters $l_i, k_1, k_2, \cdots, k_{i-1}, k_{i+1}, \cdots, k_m$ $(l_i\neq x)$. Nor is $l_i=l_1$, for if it were, $\{G(a_1), G(a_i)\}$ would have a transitive constituent of degree $km+1(<n)$ on the letters $a_1, a_2, \cdots, k_m, l_1$. Then $l_1, l_2, \cdots, l_m$ are the letters of a transitive constituent L of $G(x)$. Any substitution of $G(x)$ that replaces a_1 by a_i replaces l_1 by l_i, so that the substitutions of $G(x)$ permute the letters of A and the letters of L in exactly the same way. The groups A and L are similar.

Suppose now that $n=(k+1)m+1$. Then $G(a_1)$ has a transitive constituent X_1 on the m letters $x, l_2, \cdots, l_m$. $G(a_1)(x)$

$(=G(b_1)(x)=\cdots=G(l_1)(x))$ has $k+1$ transitive constituents on $a_2, a_3, \cdots, a_m$; $b_2, b_3, \cdots, b_m$; $\cdots$; and $l_2, l_3, \cdots, l_m$. No two of the groups $G(b_1), G(c_1), \cdots, G(l_1)$ can have the transitive constituent $a_2, a_3, \cdots$ in common. Because B_1 (on $b_1, a_2, \cdots, a_m$) is in $G(a_1)$, in one of these k groups $x, a_2, \cdots, a_m$ are the letters of a doubly transitive constituent. Let these subgroups be transformed by $(x)(a_1a_2\cdots)\cdots$ into $G(b_2), G(c_2), \cdots, G(l_2)$. One of the transformed groups has a doubly transitive constituent on the letters $x, a_1, a_3, \cdots, a_m$, and therefore contains a substitution $S=(xa_1)\cdots$ which permutes the letters $a_3, a_4, \cdots, a_m$ among themselves. Then S should transform the constituent A of $G(x)$ into the constituent X_1 of $G(a_1)$; but this, because $m\geqq 3$, S cannot do. Hence $n>(k+1)m+1$. We have shown too that $G(x)$ has $k+1$ doubly transitive constituents $A, B, \cdots, L$ whose letters are permuted by $G(x)$ in exactly the same way, and that $G(a_1)$ has k transitive constituents $B_1, C_1, \cdots, L_1$. Thus the three conditions of our assumption are reproduced with $k+1$ replacing k.

Since n is finite, this process can lead only to the conclusion that the degree of some transitive constituent of $G(x)$ is greater than m and divides $m(m-1)$.

STANFORD UNIVERSITY

A THEOREM OF KUMMER'S CONCERNING THE SECOND FACTOR OF THE CLASS NUMBER OF A CYCLOTOMIC FIELD*

BY H. S. VANDIVER

Kummer† in a letter to Kronecker dated December 28, 1849, gave without proof the result that if

$$E_n = \epsilon^l\,;\ l = 1 + sr^{-2n} + s^2r^{-4n} + \cdots + s^{(l-3)/2}r^{-(l-3)n},$$

and

$$\epsilon = \left(\frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)(1-\zeta^{-1})}\right)^{1/2}, \qquad \zeta = e^{2i\pi/l},$$

and

$$\prod_{i=1}^{(l-3)/2} E_i^{a_i} = \eta^l \tag{1}$$

where η is a unit in the cyclotomic field defined by ζ, then each $E_i^{a_i}$ is the lth power of a unit in the field, the a's being rational integers. Kummer refers to this result as follows:

"Sehr wichtiger Satz, welcher eine grosse Schwierigkeit hebt."

In the above statement the Kronecker-Hilbert‡ symbolic powers are employed; r is a primitive root of the odd prime l and s stands for the substitution (ζ/ζ^r). These symbolic powers have the following properties.§ If we denote by ω_k the integer in the field $k(\zeta)$ obtained from the integer $\omega=\omega_0$ by means of the substitution (ζ/ζ^{r^k}) then we write

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* Presented to the Society, New York, March 30, 1929.

† Abhandlungen zur Geschichte der Mathematik, vol. 29 (1910), pp. 84–85.

‡ Hilbert, *Die Theorie der Algebraischen Zahlkörper*, Berichte der Deutschen Mathematiker-Vereinigung, 1894, p. 271.

§ Landau, *Vorlesungen über Zahlentheorie*, vol. 3, pp. 253–4, proves analogous properties for a field relative to $k(\zeta)$.

$$\omega_0^{b_0}\omega_1^{b_1}\cdots\omega_{l-2}^{b_{l-2}} = \omega^{F(s)},$$

the b's being rational integers and

$$F(s) = b_0 + b_1 s + b_2 s^2 + \cdots + b_{l-2}s^{l-2}.$$

It then follows that

$$(\omega^{F(s)})^{G(s)} = \omega^{F(s)G(s)}.$$

Employing this relation we shall now prove Kummer's statement. A direct application of it gives

$$E^{s-r^{2i}} = \eta_i^{\,l},$$

and

$$E^{s^c-r^{2ci}} = \eta_{i,c}^{\,l},$$

where the η's are units in $k(\zeta)$. Applying the last equation to (1) we may write after employing the substitution s^c,

$$\prod_{i=1}^{(l-3)/2} E_i^{a_i r^{2ic}} = \xi^l,$$

where ξ is a unit in $k(\zeta)$. We obtain $(l-1)/2$ equations by taking $c=0, 1, 2, \cdots, (l-3)/2$. Raising each equation to the power r^{-2cd}, the c's varying as to the particular equation used, we then obtain

$$E_d^{a_d}\prod E_i^{a_i r^{2c(i-d)}} = \xi_c^{\,l},$$

where the product ranges over the values $i=1, 2, \cdots, (l-3)/2$ excepting d.

Multiplying the $(l-1)/2$ equations together we obtain

$$E_d^{a_d{}^{(l-1)/2}}\prod E_j^{a_j k} = \xi^l,$$

where

$$k = \frac{r^{l-1}-1}{r^{2(i-d)}-1}.$$

The numerator of this fraction is divisible by l and the denominator is prime to l since $(i-d)$ is not divisible by $(l-1)$. Hence we have

$$E_d^{a_d{}^{(l-1)/2}} = \xi_d^{\,l}.$$

Raising each side of this equation to the power l' where

$$l' \frac{l-1}{2} \equiv 1, \qquad (\text{mod } l),$$

we have Kummer's result for $d = 1, 2, \cdots, (l-3)/2$.

I think it highly probable that Kummer encountered this question in connection with a problem involving the second factor of the class number of $k(\zeta)$. I shall prove in another article that if the second factor of the class number is divisible by l, then (1) holds with not all the a's divisible by l. A somewhat similar result is proved by Hilbert.*

THE UNIVERSITY OF TEXAS

ON THE RANK EQUATION OF ANY NORMAL DIVISION ALGEBRA†

BY A. A. ALBERT‡

1. *Introduction.* The different types of normal division algebras which have been discovered up to the present depend upon equations with different groups. It has been thought that, as the rank equation of an algebra is invariant under a change of basal units, the groups of the rank equations of these various types of algebras might serve to show their non-equivalence. This notion is shown to be false here, as the group of the rank equation of any normal division algebra is the symmetric group. In proving this theorem a new theorem in the Hilbert theory of an irreducible polynomial whose coefficients are rational functions, with coefficients in any infinite field K, of several parameters is developed.

2. *General Theory.* We shall first give several presupposed

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* Loc. cit., pp. 435–437.

† Presented to the Society, December 27, 1928.

‡ National Research Fellow.

results as lemmas. Lemma 3 is an immediate consequence of Lemmas 1 and 2.

LEMMA 1. *The group of the general equation for the field K determined by its coefficients and any constants finite in number is the symmetric group.*

LEMMA 2. THE HILBERT IRREDUCIBILITY THEOREM. *Let K be any infinite field. Consider the equation*

$$(1) \quad f(x; \lambda_1, \cdots, \lambda_r) \equiv x^n + F_1(\lambda_1, \cdots, \lambda_r)x^{n-1} + \cdots + F_n(\lambda_1, \cdots, \lambda_r) = 0,$$

where $F_i(\lambda_1, \cdots, \lambda_r)$ are rational functions, with coefficients in K, of the independent parameters $\lambda_1, \cdots, \lambda_r$. Let the group of $f(x)$ with respect to $K(\lambda_1, \cdots, \lambda_r)$ be Γ. Then there exist an infinity of rational values of the parameters $\lambda_1, \cdots, \lambda_r$, such that the resulting numerical equation has the group Γ with respect to K.

LEMMA 3. *There exist an infinity of equations with leading coefficients unity and further coefficients in K such that the group of each equation with respect to K is the symmetric group.*

LEMMA 4. *Let A be the algebra of all n-rowed square matrices in K. Let*

$$(2) \quad \phi(\omega) \equiv \omega^n + \alpha_1\omega^{n-1} + \cdots + \alpha_n = 0$$

be any equation of degree n in K. Then there exists an element of A whose characteristic equation is $\phi(\omega) = 0$.

PROOF. The matrix $(\lambda_{ij} \mid i, j = 1, 2, \cdots, n)$, where

$$\lambda_{ij} = -\alpha_j, \qquad (j = 1, 2, \cdots, n),$$

$$\lambda_{i+1,i} = 1, \qquad (i = 1, 2, \cdots, n-1),$$

and all other elements are zero, has for its characteristic equation $\phi(\omega) = 0$.

As a consequence of these lemmas we have the following theorems.

THEOREM 1. *There exists an element of A, the algebra of all n-rowed square matrices with coefficients in K, whose mini-*

mum equation has degree n and the symmetric group with respect to K.

THEOREM 2. *Let* $f(x; \lambda_1, \cdots, \lambda_r) = 0$ *be an equation* (1) *with group* Γ. *Let* $\lambda_1', \cdots, \lambda_r'$ *be any set of scalars and let the resulting numerical equation*

$$(3) \quad f(x; \lambda_1', \cdots, \lambda_r') \equiv x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

have the group Γ_0 *with respect to* $K(\lambda_1', \cdots, \lambda_r')$. *Then* Γ_0 *is a sub-group of* Γ.

PROOF. Form all permutations $i_1, i_2, \cdots, i_n$ of the numbers $1, 2, \cdots, n$. Consider the product

$$P \equiv \Pi(u + x_{i_1}u_1 + \cdots + x_{i_n}u_n)$$

taken over all such permutations of the roots $x_1, \cdots, x_n$ of (1). This is a polynomial with coefficients in $K(\lambda_1, \cdots, \lambda_r)$ of the indeterminates $u, u_1, \cdots, u_n$. Let $G(u, u_1, \cdots, u_n; \lambda_1, \cdots, \lambda_r)$ be any factor of P with coefficients in $K(\lambda_1, \cdots, \lambda_r)$, irreducible in $K(\lambda_1, \cdots, \lambda_r)$. Then the group Γ of the equation $f(x; \lambda_1, \cdots, \lambda_r)$ is defined as the set of all permutations leaving G unaltered. If we replace the indeterminates $\lambda_1, \cdots, \lambda_r$ by $\lambda_1', \cdots, \lambda_r'$ as in the theorem and consider the equation $f(x; \lambda_1', \cdots, \lambda_r') = 0$, we shall obtain its group by finding all permutations leaving any factor g of $G(u, u_1, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$, having coefficients in $K(\lambda_1', \cdots, \lambda_r')$ and irreducible in $K(\lambda_1', \cdots, \lambda_r')$, unaltered. But the group of $f(x; \lambda_1', \cdots, \lambda_r')$ is unique, and hence any permutation leaving g unaltered leaves any of the other factors of $G(u, u_1, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$ unaltered and hence the product $G(u, u_1, \cdots; u_n, \lambda_1', \cdots, \lambda_r')$. Hence *all* substitutions of Γ_0 leave $G(u, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$ unaltered. The number of permutations that leave $G(u, u_1, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$ unaltered is not greater than the number leaving $G(u, \cdots, u_n; \lambda_1, \cdots, \lambda_r,$ unaltered since the degree of both polynomials in the indeterminates $u_1, \cdots, u_n$ is the same. Hence *the only* substitutions leaving $G(u, \cdots, u_n; \lambda_1', \cdots, \lambda_r')$ unaltered are substitutions of Γ.

Hence the substitutions of Γ_0 are substitutions of Γ and Γ_0 is a subgroup of Γ. This leads to the following conclusion.

THEOREM 3. *Let D be a normal division algebra in n^2 units over a field K. Then the group of the rank equation of D is the symmetric group.*

PROOF. Let $x=\sum_{i=1}^{n^2}\xi_i u_i$ be an element of D whose coordinates $\xi_1, \cdots, \xi_{n^2}$ are independent variables in K and let $R(\omega; \xi_1, \cdots, \xi_{n^2})=0$ be the rank equation of D. Then $R=0$ is an equation with leading coefficient unity and further coefficients polynomials in $\xi_1, \cdots, \xi_{n^2}$ with coefficients in K, and the degree of R is n. Let K' be an extension of K such that the algebra D', with the same basal units as D, is equivalent, by a transformation of basal elements, to the algebra of all n-rowed square matrices with coefficients in K'. This can be done by the adjunction of a finite number of constants to K.* It is also easily shown that the rank equation of D' is $R(\omega; \xi_1', \cdots, \xi_{n^2}')=0$, where $\xi_1', \cdots, \xi_{n^2}'$ are independent variables in K'.

By Theorem 1, D' contains an element y whose minimum equation is of degree n and has the symmetric group with respect to K'. Let $y=\sum_{i=1}^{n^2}\bar{\xi}_i' u_i$. Then the minimum equation of y is $R(\omega; \bar{\xi}_1', \cdots, \bar{\xi}_{n^2}')=0$. Hence for some values of $\xi_1, \cdots, \xi_{n^2}$ in K' the group of $R(\omega; \xi_1, \cdots, \xi_{n^2})$ with respect to K' is the symmetric group. Again, by Theorem 2, the symmetric group is a subgroup of the group Γ of $R(\omega; \xi_1, \cdots, \xi_{n^2})$ with respect to $K(\xi_1, \cdots, \xi_{n^2})$. Hence Γ is the symmetric group. This proves the theorem. As a corollary we have the following result.

COROLLARY. *Every normal division algebra in n^2 units over K contains an infinity of elements whose minimum equations each have degree n and the symmetric group with respect to K.*

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* L. E. Dickson, *Algebren und ihre Zahlentheorie*, p. 137.

ON THE NUMBER OF APPARENT TRIPLE POINTS OF SURFACES IN SPACE OF FOUR DIMENSIONS

BY B. C. WONG

Two hypersurfaces in 4-space of orders μ and ν respectively intersect in a surface F of order $\mu\nu$. F has a certain number H of apparent triple points, that is, lines that can be drawn through a given point meeting F three times. If F degenerates into an F_1 of order m_1 and an F_2 of order m_2 where $m_1+m_2=\mu\nu$, then H is the sum of the numbers h_{30}, h_{21}, h_{12}, h_{03}, where h_{ij} is the number of lines that pass through a given point and meet F_1 i times and F_2 j times. It is the purpose of this paper to determine H and, if F is composite, to determine the distribution of the h_{ij} lines.

The formula for H can be readily obtained by calculating the order of the restricted system of equations resulting from imposing upon two binary equations of orders μ and ν respectively the conditions that they have three common roots.* But this method does not offer a ready means for the determination of the distribution of the h_{ij} lines if F is composite. The following method seems well adapted for the purpose.

Suppose, temporarily, that the two hypersurfaces giving F be composed of μ and ν hyperplanes α_k, β_l $[k=1, 2, \cdots, \mu;$ $l=1, 2, \cdots, \nu]$. Then F is made up of $\mu\nu$ planes $\alpha_k\beta_l$. We construct the rectangular array

$$
\begin{array}{ll}
 & \alpha_1\beta_1 \quad \alpha_2\beta_1 \quad \alpha_3\beta_1 \cdots \alpha_\mu\beta_1 \\
 & \alpha_1\beta_2 \quad \alpha_2\beta_2 \quad \alpha_3\beta_2 \cdots \alpha_\mu\beta_2 \\
\text{(A)} & \alpha_1\beta_3 \quad \alpha_2\beta_3 \quad \alpha_3\beta_3 \cdots \alpha_\mu\beta_3 \\
 & \cdots\cdots\cdots\cdots\cdots \\
 & \cdots\cdots\cdots\cdots\cdots \\
 & \alpha_1\beta_\nu \quad \alpha_2\beta_\nu \quad \alpha_3\beta_\nu \cdots \alpha_\mu\beta_\nu
\end{array}
$$

* Salmon, *Modern Higher Algebra*, 4th ed., Lesson 19.

and interpret it as the symbolic representation of F, proper or improper. In this interpretation we remove the assumption that the hypersurfaces are composed of hyperplanes and the $\alpha_k\beta_l$ are to be regarded as mere symbols.

Each of the constituents of the array (A), taken alone, represents a plane. A pair of constituents represents a quadric surface or two incident planes if the constituents are in the same row as $\alpha_1\beta_1$, $\alpha_2\beta_1$ or in the same column as $\alpha_1\beta_1$, $\alpha_1\beta_2$; two non-incident planes if they are in different rows and columns from $\alpha_1\beta_1$, $\alpha_2\beta_2$. Three constituents in the same row as $\alpha_1\beta_1$, $\alpha_2\beta_1$, $\alpha_3\eta_1$ or in the same column as $\alpha_1\beta_1$, $\alpha_1\beta_2$, $\alpha_1\beta_3$ represent a cubic surface lying wholly in an S_3: if the constituents are such that one of them lies in the same column with another and in the same row with the third as $\alpha_1\beta_2$, $\alpha_1\beta_1$, $\alpha_2\beta_1$, the cubic surface is a 4-space surface. Three non-incident planes are represented by three constituents all in different rows and columns from $\alpha_1\beta_1$, $\alpha_2\beta_2$, $\alpha_3\beta_3$. Since from a given point only one line can be drawn meeting three non-incident planes each once, the presence of such a triple of constituents, all lying in different rows and columns, in the array means the presence of an apparent triple point on F. The total number of possible triples of this sort in (A) is the total number of possible apparent triple points of F and the formula for this number is evidently

$$H = \mu\nu(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2)/6. \tag{1}$$

Now if F is composed of an F_1 of order m_1 and an F_2 of order m_2, the constituents of (A) are divided into two groups: one of m_1 constituents representing F_1 and the other of m_2 constituents representing F_2. Then h_{30} is the number of triples of constituents lying in different rows and columns of the first group; h_{03} the number of similar triples in the second group; h_{21} the number of triples each consisting of a pair of constituents in the first group and one constituent in the second; h_{12} the number of triples each consisting of one in the first and two in the second. Evidently H is the sum of all the h_{ij}, that is

$$(2)\quad (m_1+m_2)(\mu-1)(\mu-2)(\nu-1)(\nu-2) = 6(h_{30}+h_{21}+h_{12}+h_{03}).$$

From the very nature of the case we have also

$$(3)\quad \begin{aligned} m_1(\mu-1)(\mu-2)(\nu-1)(\nu-2) &= a_0h_{30}+a_1h_{21}+bh_{11},\\ m_2(\mu-1)(\mu-2)(\nu-1)(\nu-2) &= a_0h_{03}+a_1h_{12}+bh_{11},\end{aligned}$$

and

$$(4)\quad (m_1-m_2)(\mu-1)(\mu-2)(\nu-1)(\nu-2) = a_0(h_{30}-h_{03})+a_1(h_{21}-h_{12}),$$

where a_0, a_1 are numerical constants, b is a function of μ and ν, and h_{11} is the order of the cone of lines through a given point meeting F_1 and F_2 each once, or the number of apparent intersections of the sections of F_1 and F_2 by an S_3. The values of a_0, a_1, being independent of μ and ν, can be determined without difficulty. If we put $m_2=0$, and consequently $m_1=\mu\nu$, $h_{03}=h_{21}=h_{12}=0$ in (4), we have $h_{30}=H$ and $a_0=6$. To determine a_1 let F_2 be of order μ, represented by a row of constituents in (A). Then $m_1=\mu\nu-\mu$, $m_2=\mu$, $h_{30}=\mu(\mu-1)(\mu-2)\cdot(\nu-1)(\nu-2)(\nu-3)/6$, $h_{21}=\mu(\mu-1)(\mu-2)(\nu-1)(\nu-2)/2$, $h_{03}=h_{12}=0$. Substituting in (4), we find $a_1=2$. To determine b, it is only necessary to make $m_2=1$. Then $h_{30}=h_{12}=0$ and $h_{11}=(\mu-1)(\nu-1)$. Substituting in the second of (3), we obtain $b=(\mu-2)(\nu-2)$. Then (3) and (4) become

$$(5)\quad \begin{aligned} m_1(\mu-1)(\mu-2)(\nu-1)(\nu-2) &= 6h_{30}+2h_{21}+(\mu-2)(\nu-2)h_{11},\\ m_2(\mu-1)(\mu-2)(\nu-1)(\nu-2) &= 6h_{03}+2h_{12}+(\mu-2)(\nu-2)h_{11},\end{aligned}$$

and

$$(6)\quad (m_1-m_2)(\mu-1)(\mu-2)(\nu-1)(\nu-2) = 6(h_{30}-h_{03})+2(h_{21}-h_{12}).$$

From (2) and (5) we obtain

$$(7)\qquad 2(h_{21}+h_{12}) = (\mu-2)(\nu-2)h_{11},$$

$$(8)\quad (m_1+m_2)(\mu-1)(\mu-2)(\nu-1)(\nu-2) = 6(h_{30}+h_{03})+3(\mu-2)(\nu-2)h_{11}.$$

From (5), (7), (8) one can calculate the h's if any two of them are known. From the divided array representing the degenerate F it is not difficult to obtain the values of two of the h's. Take a simple illustration. Let $\mu=\nu=3$ and $m_1=6$, $m_2=3$. If F_1 is symbolized by

$$\begin{array}{ccc} \alpha_1\beta_1 & \alpha_2\beta_1 & \alpha_3\beta_1 \\ \alpha_1\beta_2 & \alpha_2\beta_2 & \cdot \\ \alpha_1\beta_3 & \cdot & \cdot \end{array}$$

and F_2 by

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha_3\beta_2 \\ \cdot & \alpha_2\beta_3 & \alpha_3\beta_3 \end{array}$$

then we have, by inspection, $h_{30}=1$, $h_{03}=0$. Either from the formulas or by further inspection we see that $h_{11}=10$, $h_{21}=4$, $h_{12}=1$.

It is to be added that the method outlined above, applied to r-space, enables us to show that the number of apparent $(r-1)$-fold points of an $(r-2)$-dimensional variety which is the intersection of two hypersurfaces in S_r of orders μ and ν respectively is

$$H_{r-1} = (r-1)!\binom{\mu}{r-1}\binom{\nu}{r-1}.$$

The same process of reasoning yields the following formulas analogous to (5) and (6):

$$\frac{m_1(\mu-1)!(\nu-1)!}{(\mu-r+1)!(\nu-r+1)!} = \sum_{i=0}^{t-1} a_i h_{r-i-1,i} + \sum_{j=1}^{t} b_j h_{jj},$$

$$\frac{m_2(\mu-1)!(\nu-1)!}{(\mu-r+1)!(\nu-r+1)!} = \sum_{i=0}^{t-1} a_i h_{i,r-i-1} + \sum_{j=1}^{t} b_j h_{jj},$$

$$\frac{(m_1-m_2)(\mu-1)!(\nu-1)!}{(\mu-r+1)!(\nu-r+1)!} = \sum_{i=0}^{t-1} a_i (h_{r-i-1,i} - h_{i,r-i-1})$$

where $t=(r-1)/2$ if r is odd and $(r-2)/2$ if r is even. There is no difficulty in calculating

$$a_i = (r-2)!(r-2i-1),$$

but some difficulty is encountered in calculating b_j which are functions of μ and ν. The following are some of their values:

$$b_1 = (\mu-2)!(\nu-2)!/D,$$

$$b_2 = 4(\mu-4)!(\nu-4)!/D,$$

$$b_3 = 72(\mu-6)!(\nu-6)!/D,$$

$$b_4 = 2880(\mu-8)!(\nu-8)!/D, \quad \text{etc.},$$

where $D=(\mu-r+1)!(\nu-r+1)!$

For $r=3$, $a_0=2$, $b_1=1$ as is well known.* For $r=4$, $a_0=6$, $a_1=2$, $b_1=b=(\mu-2)(\nu-2)$ as we have seen above.

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* Salmon, *Analytic Geometry of Three Dimensions*, 5th ed., vol. 1, pp. 357, 358.

ON THE INDEPENDENCE OF THE FIRST AND SECOND MATRICES OF AN ALGEBRA*

BY C. C. MACDUFFEE

1. *Introduction.* It is well known† that every linear associative algebra with a principal unit (modulus) is isomorphic with the algebra of its first matrices, and also with the algebra of its transposed second matrices. If the algebra has no principal unit, it can be represented as a matric algebra of $(n+1)$th order matrices.

The condition that the algebra have a principal unit is not, however, necessary in order that the algebra be isomorphic with the algebra of its first or second matrices, as can readily be seen from examples. In this paper necessary and sufficient conditions for this isomorphism are obtained.

2. *The Correspondence of Poincaré.* Consider a linear associative algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ with n basal numbers $e_1, e_2, \cdots, e_n$, the constants of multiplication being c_{ijk}. Let us denote by R_i the matrix‡ (c_{ikr}), and by S_i the matrix (c_{ris}), where r determines the row and s the column in which an element stands.

The conditions for associativity in $\mathfrak{A}$ may be written§

$$\sum_k c_{ikr}c_{jsk} = \sum_k c_{ijk}c_{krs}, \quad (i,j,r,s = 1,2, \cdots, n). \tag{1}$$

If we form the matrices in which the respective members of the above equation stand in the rth row and sth column, we have

$$R_iR_j = \sum_k c_{ijk}R_k.$$

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* Presented to the Society, Chicago, March 30, 1929.

† L. E. Dickson, *Algebras and their Arithmetics*, Chicago, 1923, p. 96.

‡ R_i and S_i are the first and transposed second matrices, respectively, of e_i. Dickson, loc. cit., p. 95.

§ Dickson, loc. cit., p. 92.

We may also write conditions (1) in the form

$$\sum_k c_{rik}c_{kjs} = \sum_k c_{ijk}c_{rks},$$

whence it follows that

$$S_iS_j = \sum_k c_{ijk}S_k.$$

Thus both R_i and S_i are isomorphic under multiplication with the basal numbers e_i.

Every number a of $\mathfrak{A}$ can be written

$$a = a_1e_1 + a_2e_2 + \cdots + a_ne_n,$$

where the a_i are in $\mathfrak{F}$. We define the *first matrix* $R(a)$ of a by the equation

$$R(a) = a_1R_1 + a_2R_2 + \cdots + a_nR_n,$$

and the *second matrix* $S(a)$ by

$$S(a) = a_1S_1 + a_2S_2 + \cdots + a_nS_n.$$

Thus the algebra $\mathfrak{A}$ is isomorphic with the algebra of matrices $R(a)$ if and only if $R_1, R_2, \cdots, R_n$ are linearly independent, and isomorphic with the algebra of matrices $S(a)$ if and only if $S_1, S_2, \cdots, S_n$ are linearly independent.

3. *Two Invariants.* If we apply to the basal numbers $e_1, e_2, \cdots, e_n$ the linear transformation

$$(2) \qquad e_i = \sum_j a_{ij}e'_j, \qquad a = |a_{rs}| \neq 0,$$

with coefficients in $\mathfrak{F}$, the constants of multiplication are subject to the induced transformation*

$$(3) \qquad \sum_i c_{rsi}a_{ij} = \sum_{p,q} a_{rp}a_{sq}c'_{pqj}, \qquad (r,s,j = 1,2,\cdots,n).$$

This may be written

$$ac_{ijr} = \sum_{p,q,t} A_{rt}a_{ip}c'_{pqt}a_{sq},$$

* For example, see MacDuffee, Transactions of this Society, vol. 31 (1929), p. 81.

where A_{rt} denotes the cofactor of a_{rt} in $A=(a_{rs})$. Then

$$R_i = \bar{A}^{-1}(\Sigma_p a_{ip} R_p')\bar{A}, \qquad (i = 1, 2, \cdots, n), \tag{4}$$

where $\bar{A}$ is the transpose of A.

If we denote by

$$\sum_j k_{ij} R_j = 0, \qquad (i = 1, 2, \cdots, \rho), \tag{5}$$

a maximal set of linearly independent linear relations among the R_j, we have

$$\bar{A}^{-1} \sum_{k,j} k_{ij} a_{jk} R_k' \bar{A} = 0,$$

so that

$$\sum_k \left(\sum_j k_{ij} a_{jk} \right) R_k' = 0, \qquad (i = 1, 2, \cdots, \rho).$$

Since

$$\left(\sum_j k_{rj} a_{js} \right) = (k_{rs})A,$$

and the matrix (k_{rs}) of ρ rows and n columns is of rank ρ, we see that there are at least ρ linearly independent linear relations among the matrices R_1', R_2', $\cdots$, R_n'. Since (2) has an inverse, there are just ρ such relations. Hence ρ is invariant under transformation of coordinates.

Similarly we find that the number σ of linearly independent linear relations among the matrices S_1, S_2, $\cdots$, S_n is likewise invariant.

4. *A Condition for the Independence of the Matrices.* Suppose that exactly ρ independent relations (5) hold among the matrices R_j. Form a matrix $B \equiv (b_{rs})$ so that $b_{ij} = k_{ij}$ for $i = n-\rho+1, \cdots, n$, and take for the remaining b_{ij} any convenient numbers of $\mathfrak{F}$ so that B is non-singular. Apply a transformation (2) using $A = B^{-1}$. From (4) we have

$$R_i' = \bar{B}^{-1} \sum_j b_{ij} R_j \bar{B}, \qquad (i = 1, 2, \cdots, n).$$

Hence $R'_{n-\rho+1} = \cdots = R'_n = 0$ while $R'_1, R'_2, \cdots, R'_{n-\rho}$ are linearly independent. We drop primes.

We now have

$$c_{ijk} = 0, \qquad (i > n-\rho\,;\ j,k = 1,2,\cdots,n). \tag{6}$$

The associativity conditions (1) may be written

$$\sum_{k=1}^{n} c_{ijk}c_{ksr} = \sum_{k=1}^{n} c_{ikr}c_{jsk}.$$

We consider only those equations in which $j > n-\rho$, and pass to matrices, obtaining

$$\sum_{k=1}^{n-\rho} c_{ijk}R_k = 0.$$

Since $R_1, R_2, \cdots, R_{n-\rho}$ are linearly independent,

$$c_{ijk} = 0, \quad (j > n-\rho\,;\ k \leqq n-\rho\,;\ i = 1,2,\cdots,n). \tag{7}$$

Consider the linear set $\mathfrak{Z}$ composed of all numbers

$$z = z_{n-\rho+1}e_{n-\rho+1} + \cdots + z_n e_n.$$

We see readily from (6) and (7) that

$$\mathfrak{Z}\mathfrak{A} = 0, \qquad \mathfrak{A}\mathfrak{Z} \leqq \mathfrak{Z}, \tag{8}$$

so that $\mathfrak{Z}$ is an invariant zero subalgebra of $\mathfrak{A}$ which has the additional property that $\mathfrak{Z}\mathfrak{A} = 0$.

Conversely, let us suppose that $\mathfrak{A}$ has an invariant zero subalgebra $\mathfrak{Z}$ of order ρ such that $\mathfrak{Z}\mathfrak{A} = 0$. We take the basis numbers of $\mathfrak{Z}$ for $e_{n-\rho+1}, \cdots, e_n$ of a basis for $\mathfrak{A}$. Since $\mathfrak{Z}\mathfrak{A} = 0$, we have (7) and therefore $R_{n-\rho+1} = \cdots = R_n = 0$.

Similar results hold for the second matrix.

THEOREM 1. *A necessary and sufficient condition in order that there be exactly $\rho(\sigma)$ linearly independent linear relations among the matrices $R_1, R_2, \cdots, R_n$ $(S_1, S_2, \cdots, S_n)$ is that $\mathfrak{A}$ have an invariant zero subalgebra $\mathfrak{Z}(\mathfrak{W})$ of order $\rho(\sigma)$ such that $\mathfrak{Z}\mathfrak{A} = 0$, $(\mathfrak{A}\mathfrak{W} = 0)$, and no such subalgebra of order greater than $\rho(\sigma)$.*

It is known that in every algebra with a principal unit (including every semi-simple algebra) both the first and second matrices are independent.*

It can be seen directly that if $\mathfrak{A}$ is nilpotent or the direct sum of a nilpotent algebra and another algebra, the first matrices and also the second matrices are dependent. For we may choose normalized† basis numbers e_i for the nilpotent algebra such that

$$e_i e_n = e_n e_i = 0, \qquad (i = 1, 2, \cdots, n).$$

Hence e_n serves both as $\mathfrak{Z}$ and $\mathfrak{W}$ for Theorem 1.

That ρ is not necessarily equal to σ is shown by the following example:

$$e_1^2 = e_1, \qquad e_1 e_2 = 0, \qquad e_2 e_1 = e_2, \qquad e_2^2 = 0.$$

We have

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here R_1 and R_2 are linearly independent and give a representation of the algebra, while S_1 and S_2 are dependent. Thus $\rho = 0$, $\sigma = 1$, and $\mathfrak{W}$ is composed of multiples of e_2.

5. *The Characteristic Equations.* Let

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

be the general number of an algebra $\mathfrak{A}$, and define

$$\delta(\omega) = \left| R(x) - \omega I \right|, \qquad \delta'(\omega) \equiv \left| S(x) - \omega I \right|.$$

It is well known‡ that $x\delta(x) = 0$ and $x\delta'(x) = 0$, and if $\mathfrak{A}$ has a principal unit, also $\delta(x) = 0$ and $\delta'(x) = 0$.

It is possible, however, for $\delta(x) = 0$ or $\delta'(x) = 0$ even though $\mathfrak{A}$ has no principal unit.

THEOREM 2. *A sufficient condition in order that* $\delta(x) = 0$ $(\delta'(x) = 0)$ *is that* $R_1, R_2, \cdots, R_n$ $(S_1, S_2, \cdots, S_n)$ *be linearly independent.*

* Dickson, loc. cit., p. 95.

† Dickson, loc. cit., p. 176.

‡ Dickson, loc. cit., p. 100.

The proof is immediate, for by the Hamilton-Cayley theorem

$$\delta(R(x)) = 0, \qquad \delta'(S(x)) = 0.$$

Since $\mathfrak{A}$ is isomorphic with the algebra of matrices $R(x)$ (or $S(x)$), we have $\delta(x)=0$ (or $\delta'(x)=0$).

For the example of §4 we have

$$\delta(\omega) = \omega^2 - \omega x_1, \qquad \delta'(\omega) = \omega^2 - 2\omega x_1 + x_1^2.$$

Hence $\delta(x)=0$, while $\delta'(x)=x_1^2-x_1^2e_1-x_1x_2e_2$.

OHIO STATE UNIVERSITY

ON THE NUMBER $(10^{23}-1)/9$

D. H. LEHMER

The purpose of this note is to save any further effort* in trying to factor the number $N=(10^{23}-1)/9=111,\ 11111,\ 11111,\ 11111,\ 11111$ which in a previous paper was found to be composite.† This assertion was based on a negative result giving $3^{N-1}\not\equiv 1 \pmod N$.

On the basis of this conclusion Kraitchik‡ attempted to factor N arriving at another negative result that N had no factors and therefore was a prime. This conflict of results led us to recompute the value of $3^{N-1} \pmod N$ which shows clearly a mistake in the original calculation arising from the choice of 3 for a base instead of another number prime to $10^{23}-1$. Such another base would have furnished the extra check which would have detected the error.

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* A recent letter from Mr. R. E. Powers informs us that he has been to the trouble of finding 150 quadratic residues of N.

† This Bulletin, vol. 33 (1927), p. 338.

‡ Mathesis, vol. 42 (1928), p. 386.

The recomputation revealed the following results:

$$3^{N-1} \equiv 1 \pmod{N},$$

$$3^{(N-1)/11} \equiv 14\ 45009\ 64787\ 71867\ 25049 = r_1 \pmod{N},$$

$$3^{(N-1)/4093} \equiv 98\ 37816\ 77563\ 73768\ 37434 = r_2 \pmod{N},$$

$$((r_1 - 1), N) = ((r_2 - 1), N) = 1.$$

By Theorem 3 of my paper cited above, it follows that the factors of N belong to the forms

$$\left.\begin{matrix} 23n + 1 \\ 121n + 1 \\ 4093n + 1 \end{matrix}\right\} 11390819n + 1.$$

If we seek to express N as the difference of squares $(a^2 - b^2)$, we have

$$a = 129750757490761n + 115222895547343.$$

If we restrict a modulo 12 and 25, the smallest admissible value to try is

$$a = 5435003952668544.$$

The total range for a is given by the inequalities

$$N^{1/2} < a < \frac{1}{2}\left(W + \frac{N}{W}\right),$$

where $W = 22781638$, that is,

$$a < 243861122499491.$$

The maximum value of a is less than the smallest possible value; therefore a does not exist and N is a prime.

The results of Kraitchik's investigations will occupy a whole chapter of his forthcoming book.* Those interested in the factorization of large numbers will await with interest the exposition of the method by which Kraitchik was able to identify this sixth largest prime known.

BROWN UNIVERSITY

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* *Recherches sur la Théorie des Nombres*, vol. 2.

ON THE ATTRACTION OF SPHERES IN ELLIPTIC SPACE*

BY JAMES PIERPONT

1. *Introduction.* C. Neumann, Killing, and Liebmann have treated the motion of a material particle about a center of attraction in elliptic (hyperbolic) space. The question arises do these results hold when the center of attraction is replaced by a spherical mass.

Let the sphere be placed at the origin of coordinates O, let the polar coordinates of an element of volume at P of the sphere be ρ, ϕ, θ, where ϕ, θ are latitude and longitude. The element of volume is then

$$dv = \sin^2 \rho \cos \phi d\theta d\rho d\phi,$$

where for simplicity we take the space constant $R=1$. We will suppose the elementary mass A attracted by the sphere is on the z axis. Let $OA=\alpha$, $AP=\epsilon$, in elliptic measure. The force of attraction we will take to be

$$F = \frac{cdv}{\sin^2 \epsilon}, \qquad c \text{ a constant.}$$

If ψ is the angle AP makes with the z axis, the work done by the force F for a small displacement of A of extent $\delta\alpha$ along the z axis is

$$\delta W = F \cos \psi dv \cdot \delta\alpha.$$

It will be convenient to set

$$a = \sin \alpha, \quad r = \sin \rho, \quad e = \sin \epsilon, \quad p = \sin \phi,$$

$$a' = \cos \alpha, \quad r' = \cos \rho, \quad e' = \cos \epsilon, \quad p' = \cos \phi.$$

We have then

* Presented to the Society, New York, March 29, 1929.

$$\cos\psi = \frac{(r' - a'e')}{ae},$$

$$e = (\alpha p^2 + \beta p + \gamma)^{1/2},$$

where

$$\alpha = -a^2r^2, \qquad \beta = -2aa'rr', \qquad \gamma = 1 - a'^2r'^2.$$

Then

$$\delta W = 2\pi c(aG - a'H)\delta\alpha,$$

where

$$G = \int_0^\rho r^2r'L d\rho, \qquad L = \int_{-\pi/2}^{\pi/2} p'd\phi/e^3,$$

$$H = \int_0^\rho r^3M d\rho, \qquad M = \int_{-\pi/2}^{\pi/2} pp'd\phi/e^3.$$

Now

$$L = \left[\frac{4\alpha p + 2\beta}{\Delta(\alpha p^2 + \beta p + \gamma)^{1/2}}\right]_{-1}^{1}, \qquad \Delta = 4\alpha\gamma - \beta^2 = -4a^2r^2,$$

$$\alpha + \beta + \gamma = 1 - (a'r' + ar)^2,$$

$$(\alpha + \beta + \gamma)^{1/2} = \sin(\alpha - \rho),$$

$$(\alpha - \beta + \gamma)^{1/2} = \sin(\alpha + \rho),$$

$$4\alpha + 2\beta = -4ar\cos(\alpha - \rho),$$

$$4\alpha - 2\beta = 4ar\cos(\alpha + \rho).$$

Hence

$$L = \frac{2}{ar}\,\frac{\sin 2\rho}{\cos 2\rho - \cos 2\alpha}, \qquad G = \frac{1}{a}\int_0^\rho \frac{\sin^2 2\rho d\rho}{\cos 2\rho - \cos 2\alpha}.$$

Turning to M, we have

$$M = -\left[\frac{4\gamma + 2\beta p}{\Delta(\alpha p^2 + \beta p + \gamma)^{1/2}}\right]_{-1}^{1},$$

$$4\gamma - 2\beta = 4(1 - a'r'\cos(\alpha + \rho)),$$

$$4\gamma + 2\beta = 4(1 - a'r'\cos(\alpha - \rho)).$$

Hence

$$M = \frac{M'}{a^2r^2} + \frac{a'r'}{a^2r^2}M'',$$

where

$$M' = \frac{-4\sin\rho\cos\alpha}{\cos 2\alpha - \cos 2\rho}, \quad M'' = \frac{-2\sin 2\rho}{\cos 2\rho - \cos 2\alpha}.$$

Hence

$$M = \frac{2a'}{a^2}\cdot\frac{r}{a^2 - r^2}.$$

Thus

$$G = \frac{2}{a}\int_{-1}^{1}\frac{r^2(1-r^2)^{1/2}}{a^2 - r^2}dr,$$

$$H = \frac{2a'}{a^2}\int_{-1}^{1}\frac{r^4dr}{T},$$

where $T = (a^2 - r^2)(1 - r^2)^{1/2}$. Then

$$aG - a'H' = 2\int_0^r \frac{r^2(1-r^2)dr}{T} - \frac{2}{a^2}\int_0^r \frac{r^2dr}{T}$$

$$= \frac{2}{a^2}\int_0^r \frac{r^2dr}{(1-r^2)^{1/2}} = \frac{1}{2a^2}(2\rho - \sin 2\rho).$$

Now the volume of a sphere of radius ρ in elliptic measure is $V = \pi(2\rho - \sin 2\rho)$. As $m = cV$ is the mass of the sphere, this gives

$$\delta W = \frac{m}{a^2}\cdot\delta\alpha.$$

The attraction of the sphere on the point A is thus m/a^2, which is the same as if the mass of the sphere were concentrated at its center.

2. *Another Method.* Let us define the potential of an element of mass dm at a distance ϵ by $dU = c\,\mathrm{ctn}\,\epsilon\cdot dv$, c constant, and for the whole body supposed homogeneous, by

$$U = c\int \mathrm{ctn}\,\epsilon\cdot dv.$$

Then α being as in §1,

$$\frac{\partial U}{\partial \alpha} = c \int dv \cdot \frac{\partial}{\partial \alpha} \operatorname{ctn} \epsilon = c \int dv \cdot \frac{\partial \operatorname{ctn} \epsilon}{\partial \epsilon} \cdot \frac{\partial \epsilon}{\partial \alpha}.$$

Let A' be the point on the z axis at a distance $\alpha+\delta\alpha$ from O. Let B lie on PA' at a distance ϵ from P. Then in the small triangle ABA' we have

$$\cos \psi = \frac{\sin BA'}{\sin AA'}, \quad \text{or} \quad \cos \psi = \frac{\partial \epsilon}{\partial \alpha}.$$

Thus

$$-\frac{\partial U}{\partial \alpha} = c \int \frac{\cos \psi}{\sin^2 \epsilon} \cdot dv = \int F \cos \psi \cdot dv = \int F_z \, dv,$$

where F_z is the component of F along the z axis.

This result gives us another method of finding the attraction of a homogeneous sphere on a point at A. We have

$$U = c \int \operatorname{ctn} \epsilon \, dv = c \int \frac{e'}{e} r^2 p' d\theta d\rho d\phi$$

$$= 2\pi c(a'G_1 + aH_1),$$

where

$$G_1 = \int_0^\rho r^2 r' L_1 d\rho, \qquad L_1 = \int_{-1}^{1} \frac{dp}{Q}, \; Q = (\alpha p^2 + \beta p + \gamma)^{1/2},$$

$$H_1 = \int_0^\rho r^3 M_1 d\rho, \qquad M_1 = \int_{-1}^{1} \frac{p dp}{Q}.$$

Now

$$L_1 = \frac{1}{(-\alpha)^{1/2}} \left\{ \sin^{-1} \frac{\beta - 2\alpha}{(-\Delta)^{1/2}} - \sin^{-1} \frac{\beta + 2\alpha}{(-\Delta)^{1/2}} \right\},$$

$$\beta - 2\alpha = -2ra(a'r' - ar) = -2ar \cos(\alpha + \rho),$$

$$\beta + 2\alpha = -2ar \cos(\alpha - \rho).$$

Hence

$$L_1 = \frac{1}{ar} \left\{ \sin^{-1}(\cos(\alpha + \rho)) - \sin^{-1}(\cos(\alpha - \rho)) \right\} = \frac{2\rho}{ar}.$$

Similarly

$$M_1 = \left[\frac{(\alpha\rho^2 + \beta\rho + \gamma)^{1/2}}{\alpha}\right]_{-1}^{1} - \frac{\beta L_1}{2\alpha}$$

$$= \frac{1}{\alpha}\left\{(\alpha+\beta+\gamma)^{1/2} - (\alpha-\beta+\gamma)^{1/2}\right\} - \frac{\beta L_1}{2\alpha}$$

$$= \frac{1}{a^2r^2}\left\{\sin(\alpha+\rho) - \sin(\alpha-\rho)\right\} - \frac{2a'r'\rho}{a^2r^2} = \frac{2a'}{a^2r^2}(r - r'\rho).$$

Hence

$$G_1 = \frac{1}{4a}(\sin 2\rho - 2\rho\cos 2\rho),$$

$$H_1 = \frac{2a'}{a^2}(\tfrac{1}{2}\rho - \tfrac{1}{4}\sin 2\rho - \tfrac{1}{8}\sin 2\rho + \tfrac{1}{4}\rho\cos 2\rho).$$

Thus

$$U = \frac{a'}{2a}(2\rho - \sin 2\rho) = \frac{a'm}{a} = m\,\mathrm{ctn}\,a\,;$$

that is, the potential of a sphere at A is the same as if the whole mass were concentrated at its center.

3. *Attraction at a Point Within.* We consider now the potential W of a homogeneous spherical shell at a point within the cavity. This may be treated briefly. Proceeding as in §2, we have

$$W = 2\pi c(a'G_2 + aH_2),$$

where

$$G_2 = \int r^2 r' L_2 d\rho, \qquad H_2 = \int r^3 M_2 d\rho.$$

Here L_2, M_2 are the same as L_1, M_1 in §2 except that now we must take $\rho > \alpha$. Hence

$$L_2 = \frac{2\alpha}{ar} = -ar, \qquad M_2 = \frac{2r'}{ar^2} + a'r'.$$

These give

$$G_2 = -a\int_{\rho_1}^{\rho_2} r^2 dr, \qquad H_2 = \frac{2}{a}\int_{\rho_1}^{\rho_2} r dr + a'\int_{\rho_1}^{\rho_2} r^2 dr.$$

Hence

$$W = 4\pi c\int_{\rho_1}^{\rho_2} r dr$$

which is a constant. Thus the attraction on a point within the cavity is null.

4. *Conclusion.* We have supposed the sphere to be of uniform density. It is obvious from the above that we get the same result if the sphere is made up of concentric layers each of constant density, or if the density is a function of ρ. Finally we remark that the analysis employed may be extended readily to hyperbolic space.

In connection with the present note we wish to call attention to a valuable paper* contributed by F. Cajori to the memorial volume "Sir Isaac Newton 1727–1927." It is almost universally held that Newton's delay was due to the fact that the earth's radius was not correctly known at that epoch. Cajori argues conclusively, so we believe, that the real reason was much deeper. It was in fact Newton's inability at that time to show that the attraction of a sphere on an external particle is the same as if the mass of the sphere were concentrated at its center.

YALE UNIVERSITY

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* *Newton's twenty year's delay in announcing the law of gravitation.*

THE EXISTENCE OF THE LEBESGUE-STIELTJES INTEGRAL*

BY R. L. JEFFERY

A definition of a Lebesgue-Stieltjes integral of a function $f(x)$ defined on (a, b) with respect to a non-decreasing function $V(x)$ bounded on (a, b) has been given by Hildebrandt.†

This definition involves the idea of the measurability of f with respect to V. If α is the interval $a'<x<b'$, then $V(\alpha)=V(b'-0)-V(a'+0)$. Let a set E be enclosed in a finite or countably infinite set of non-overlapping open intervals $A\equiv\alpha_1, \alpha_2, \cdots$. Let $V(E)$ be the lower limit of $V(A)=\sum V(\alpha_i)$ for all possible enclosures A. In the same way define $V(CE)$. When

$$V(E) + V(CE) = V(a,b) = V(b) - V(a), \tag{1}$$

the set E is said to be measurable relative to V. If for all real values of l the set for which $f>l$ satisfies (1), then f is measurable relative to V. Hobson‡ gives a definition which involves a different formulation of the same idea. To state this we shall make use of the following correspondence between the points of $\alpha=V(a)\leqq u\leqq V(b)=\beta$ and $a\leqq x\leqq b$. First, if x is a point of discontinuity of V, then x goes by means of $u=V(x)$ into the closed interval $V(x-0)\leqq u \leqq V(x+0)$. There will then correspond to each u on (α, β) at least one value of x on (a, b). If to a value of u there corresponds more than one value of x, then V is constant throughout an interval, and x_u shall be the lower end point of this interval, or the lower bound of points of the interval in case it is open. If $f(x)$ is any function defined on (a, b), then $\psi(u)$ is defined by $\psi(u)=f(x_u)$ and

$$LS\int_a^b f(x)dV(x) = L\int_\alpha^\beta \psi(u)du,$$

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* Presented to the Society, September 7, 1928.

† This Bulletin, vol. 24, pp. 188–190.

‡ *Theory of Functions of a Real Variable*, 3d ed., vol. I, §445.

when the latter exists. Thus the requirement that f be measurable relative to V is replaced by the requirement that ψ be measurable on (α, β). In this note we determine some necessary and sufficient conditions bearing on V under which all Lebesgue measurable functions f are measurable relative to V. These conditions are also necessary and sufficient to insure that all measurable functions f are carried into functions $\psi(u) = f(x_u)$ measurable on (α, β).

As a matter of notation, we shall use A and B to indicate sets of intervals enclosing E and CE respectively. Each set shall be non-overlapping and open, and each interval shall contain at least one point of the set enclosed. E_u shall denote the set into which E_x is carried by means of the relation $u = V(x)$, a point of discontinuity of V going into a closed interval as agreed upon above. If $x_1, x_2, \cdots$ are the countable set of discontinuities of V and s_i the saltus of V at x_i, then, for a given ϵ, $K_x \equiv x_1, x_2, \cdots, x_k$ where $\sum_{i=k+1}^{\infty} s_i < \epsilon/2$. For convenient reference we state some properties* of all sets G which are measurable relative to V.

(1) If E_1 and E_2 belong to G then $E_1 + E_2$ belongs to G.

(2) The Borel measurable sets belong to G.

(3) A necessary and sufficient condition that E be measurable relative to V is that there exist intervals A and B enclosing E and CE such that $V(AB) < \epsilon$.

(4) Any set E on a set of open intervals throughout which V is constant is measurable relative to V.

This follows at once from (3). For evidently E can be put in A where $V(A) = 0$. If then CE is put in any set B, we have $0 \leqq V(AB) \leqq V(A)$, which shows that (3) is satisfied.

THEOREM I. *A necessary and sufficient condition that E_x be measurable relative to V is that E_u be measurable on (α, β).*

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* (1) and (2) are proved by Radon, Wiener Sitzungberichte, vol. 122, pp. 1305 ff. (3) is proved by Bliss, (this Bulletin, vol. 24, p. 13), for continuous monotone functions, but the proof is applicable to any non-decreasing function.

Suppose E_x measurable relative to V. Then by (3) we can determine A_x and B_x so that $V(A_xB_x)<\epsilon$. The open intervals A_x go, by means of $u=V(x)$, into a set of open intervals A_u together with a countable set of points A_u' which belong to E_u, and E_u is on A_u+A_u'. Similarly CE_u is on open intervals B_u and a countable set of points B_u'. But $m(A_uB_u)=V(A_xB_x)<\epsilon$. This, together with the fact that A_u' and B_u' are countable, shows that E_u is measurable.

Suppose E_u is measurable. Let $\overline{E}_u$ be the closed intervals of E_u which come from points of discontinuity of V. $\overline{E}_x$ is then a countable set of points which, on account of (2) is measurable relative to V. Let $e_u=E_u-\overline{E}_u$. If K_u is the set of closed intervals coming from K_x, then e_u is on the open intervals CK_u. Put e_u in A_u and ce_u in B_u where A_u is on CK_u and where $m(A_uB_u)<\epsilon/2$. The open intervals A_u and B_u correspond to open intervals A_x and B_x which contain e_x and ce_x respectively. And since A_u is not on K_u, A_x contains none of the points of the set K_x. Consequently (A_xB_x) contains none of the set K_x, and as a result of this

$$V(A_xB_x) \leqq m(A_uB_u) + \sum_{i=k+1}^{\infty} s_i < \epsilon.$$

It then follows from (3) that e_x is measurable relative to V, and by (1) so also is $E_x=\overline{E}_x+e_x$.

It has been shown by Carathéodory* that the function V can be represented as the sum of two functions $V=\phi+\chi$, where ϕ depends only on the discontinuities of V and where χ is continuous. We can now prove the following theorem.

THEOREM II.† *In order that every measurable set E_x go into a measurable set E_u, it is necessary and sufficient that χ be absolutely continuous.*

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* *Vorlesungen über Reelle Funktionen*, §153.

† The corresponding theorem for continuous functions has been proved by Rademacher, (Monatshefte für Mathematik und Physik, vol. 27 (1916), p. 266). His theorem is the special case of ours where ϕ is identically zero.

The condition is sufficient. Suppose E_x measurable. Let ${}_1E_x$ be the part of E_x which belongs either to the discontinuities of V or to intervals throughout which V is constant. ${}_1E_u$, being a set of closed intervals and a countable set of points, is measurable. We complete the argument by showing that $e_u = E_u - {}_1E_u$ can be put in A_u and ce_u in B_u, where $m(A_uB_u) < \epsilon$. The function χ being absolutely continuous, there exists a $\delta > 0$ such that if D is any set of intervals with $m(D) < \delta$, then $\chi(D) < \epsilon/2$. The set e_x contains none of the points of K_x. Hence, since e_x is measurable and since K_x is a finite set, e_x can be put in A_x and ce_x in B_x, where A_x contains none of the points of K_x, and where $m(A_xB_x) < \epsilon/2$. Since each interval of the set A_x contains at least one point of e_x, and since the end points of B_x are points of e_x it follows that A_x and B_x correspond to open intervals A_u and B_u containing e_u and ce_u respectively. Again, since A_x contains none of the points of K_x, neither does (A_xB_x). Hence,

$$m(A_uB_u) = V(A_xB_x) = \phi(A_xB_x) + \chi(A_xB_x) < \sum_{i=k+1}^{\infty} s_i + \epsilon/2 < \epsilon.$$

The condition is also necessary. Suppose χ is not absolutely continuous. For a given $\epsilon > 0$ let $\chi(\delta)$ be the upper limit of $\chi(\alpha)$ for all possible sets of intervals α for which $m\alpha < \delta$. Obviously, if $\delta_1 < \delta_2$, then $\chi(\delta_1) \leqq \chi(\delta_2)$. And since χ is not absolutely continuous, $\lim_{\delta \to 0} \chi(\delta) = d > 0$. Making use of these facts we now prove the following lemma.

LEMMA I. *There exists on (a, b) a set of points D_x of zero measure which contains no points of discontinuity of V, and no points of intervals throughout which V is constant, and which is carried, by means of $u = V(x)$, into a measurable* set D_u with $mD_u \geqq d$.*

Choose $\delta_1, \delta_2, \cdots$, an infinite sequence of positive numbers approaching zero monotonically, and such that $\sum \delta_i$

* Carathéodory (loc. cit., §512) has considered the analogous problem for continuous monotone functions. By a method different from ours he arrives at the existence of a set D_x of zero measure which goes into a set D_u with outer measure $\geqq d$.

converges. Since $\chi(\delta_i) \geqq d$ for all i, we can find a set of intervals α_x^i such that $\chi(\alpha_x^i) > d - \epsilon$, and $m\alpha_x^i < \delta_i$. The infinite sequence of sets of intervals α_x^i defines, by means of $u = \chi(x)$, an infinite sequence of sets of intervals α_u^i on $\alpha' = \chi(a) \leqq u \leqq \beta = \chi(b)$, where $m\alpha_u^i = \chi(\alpha_x^i) > d - \epsilon$, $(i = 1, 2, \cdots)$. It then follows* that on (α', β') we have a measurable set of points D_u'' where $mD_u'' > d - \epsilon$, and each point of which belongs to an infinite number of the sets of intervals α_u^i. If we remove from D_u'' the countable set which comes by means of $u = \chi(x)$ from the countable set of discontinuities of V, there remains a set D_u'' which still has measure $> d - \epsilon$. Let D_x'' be the set on (a, b) whose image, by means of $u = \chi(x)$, is D_u''. Each point of D_u'' is on an infinite set of the sets of intervals α_u^i. Consequently each point of the set D_x'' is on an infinite set of the sets of intervals α_x^i. Thus, for $i = l$, where l is as large as we wish, D_x'' is on an infinite set of the intervals $\alpha_x^{l+1}, \alpha_x^{l+2}, \cdots$. Hence $mD_x'' \leqq \sum_{i=l+1}^{\infty} m\alpha_x^i \leqq \sum_{i=l+1}^{\infty} \delta_i$. On account of the convergence of $\sum \delta_i$ and the fact that we can take l as great as we please, it follows that $mD_x'' = 0$. If now we consider a sequence of values of ϵ approaching zero, we get a sequence of sets ${}_1D_x'', {}_2D_x'', \cdots$, where ${}_iD_x''$ has zero measure and ${}_iD_u''$ has measure $\geqq d - \epsilon_i$. It follows that the set D_u' which consists of all the points in any of the sets ${}_iD_u'$ has measure $\geqq d$, and the corresponding set D_x' has measure zero.

It will now be shown that $u = V(x)$ carries D_x' into a set $\overline{D}_u$ measurable on (α, β) with $m\overline{D}_u \geqq d$. On (α', β'), D_u' is measurable and contains none of the points K_u which come from K_x by means of $u = \chi(x)$. Hence D_u' can be put in A_u and CD_u' in B_u where A_u contains none of the finite set K_u, and where $m(A_uB_u) < \epsilon/2$. The intervals A_u and B_u correspond by means of $u = \chi(x)$ to intervals A_x and B_x, where A_x contains D_x' and contains none of the points of K_x,

* W. H. Young, Proceedings of the London Mathematical Society, (2), vol. 2, p. 26. Also Borel, Comptes Rendus, December, 1903.

and where $\chi(A_xB_x)=m(A_uB_u)\leqq\epsilon/2$. But since A_x contains no points of K_x neither does (A_xB_x). Hence

$$V(A_xB_x)=\phi(A_xB_x)+\chi(A_xB_x)\leqq\sum_{i=k+1}^{\infty}s_i+\frac{\epsilon}{2}<\epsilon.$$

Hence D_x' is measurable relative to V. It then follows from Theorem I that $\overline{D}_u$ is measurable on (α,β). And evidently $m\overline{D}_u$ cannot be less than $\chi(D_x')\geqq d$. If we now remove from $\overline{D}_u$ the countable set which comes by means of $u=V(x)$ from intervals throughout which V is constant, we arrive at a set D_u, where $mD_u\geqq d$, and where the corresponding set D_x contains no discontinuities of V and no points of intervals throughout which V is constant, and where $mD_x=0$.

If N_u is any non-measurable component of D_u, then N_x is part of D_x and $mN_x=0$. But $u=V(x)$ carries N_x into N_u. Hence V does not carry every set measurable on (a,b) into a set measurable on (α,β). If $f(x)$ is a function such that $f=1$ on N_x and zero elsewhere on (a,b), then $\psi(u)=f(x_u)=1$ on N_u and zero elsewhere on (α,β). Thus $\psi(u)$ is not measurable on (α,β). Also, by Theorem I, f is not measurable relative to V. We have the following result.

THEOREM III. *A necessary and sufficient condition that every function f measurable on (a,b) be measurable relative to V is that χ be absolutely continuous. This condition is also necessary and sufficient that $\psi(u)=f(x_u)$ be measurable on (α,β).*

THEOREM IV.* *A necessary and sufficient condition that every function f bounded and measurable on (a,b) possess a Lebesgue-Stieltjes integral in the sense of Hildebrandt, or in the sense of Hobson, is that χ be absolutely continuous.*

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* M. D. Menchoff and Mlle. Bary, Annali di Mathematica, (4), vol. 5, have considered the Lebesgue-Stieltjes integral of f with respect to V where V is absolutely continuous. They show (p. 24) that the integral according to their definition exists whenever f is bounded and measurable. Their discussion is only applicable to the case where V is absolutely continuous.

ZERO-FREE REGIONS OF LINEAR PARTIAL FRACTIONS*

BY MORRIS MARDEN†

1. *Introduction.* The object of this paper is to determine simple regions in the plane which do not contain any zeros of the partial fraction

$$\Phi = \sum_{j=1}^{n} \frac{\alpha_j}{z - a_j},$$

when the α_j's are complex constants.

The case of real α_j's has already been adequately treated by Gauss, Lucas, Jensen, and Bôcher in connection with their study of the derivative of a polynomial and the jacobian of two binary forms. In terms of Φ, their results may be stated as follows.

(a) If all the α_j's have the same sign, there are no zeros of Φ outside of any convex polygon enclosing the points a_j. If all the a_j, in addition, lie on the line-segment AB, there are no zeros of Φ except at points on AB‡.

(b) If all the α_j's have the same sign, and the a_j's are either real or in conjugate imaginary pairs, there are no imaginary zeros of Φ at points outside of all the Jensen circles.‖ In any Jensen circle containing k a_j's, and not overlapping any other Jensen circle, there are at least $k-1$ and at most $k+1$ roots of Φ.¶

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* Presented to the Society, October 27, 1928.

† National Research Fellow, at Princeton University.

‡ The result of which this theorem is an immediate consequence was first stated by Gauss, (Werke, vol. 3, p. 112), in 1816, but rediscovered by Lucas (Comptes Rendus, 1868). This theorem, with reference to the derivative of a polynomial, was first stated by Lucas, Journal de l'École Polytechnique, vol. 46 (1879), p. 8.

‖ See §6 for definition. This part of the theorem was stated without proof by Jensen, Acta Mathematica, vol. 36 (1912), p. 190; proved by Walsh, Annals of Mathematics, vol. 22 (1920), pp. 128–144.

¶ Walsh, ibid.

(c) If $\sum\alpha_j=0$ and if all the a_j corresponding to positive α_j and to negative α_j lie respectively in the non-overlapping circular regions C_1 and C_2, there are no zeros of Φ at points outside of these regions.*

2. *Method.* In the case of complex α_j's as in the case of the real, the zero-free regions of Φ may be readily obtained when $K(\Phi)$ the conjugate imaginary of Φ, is interpreted as the resultant of all the vectors†

$$f_j = \frac{\beta_j}{\bar{z} - \bar{a}_j} \quad \text{and} \quad g_j = -\frac{i\gamma_j}{\bar{z} - \bar{a}_j}; \qquad (j = 1,2, \cdots, n).$$

where $\beta_j+i\gamma_j=\alpha_j$. If we set $a_j-z=r_je^{i\theta j}$, we may write

$$f_j = \frac{\beta_j}{r_j}e^{i(\theta_j+\pi)} \quad \text{and} \quad g_j = \frac{\gamma_j}{r_j}e^{i(\theta_j+\pi/2)}.$$

These equations tell us that, as vectors drawn from z, the first along the line a_jz and the second perpendicular to this line, and having magnitudes contingent only upon r_j and α_j, the f_j and g_j are both independent of choice of axes. A zero-free region of Φ is then one in which the vector

$$K(\Phi) = \sum_{j=1}^{n}(f_j + g_j)$$

or its component in any direction does not vanish.

As the α_j are here complex numbers, it is convenient to regard them as points in a separate α-plane. In this plane it may be assumed without loss of generality, due to the form of Φ, that none of the α_j's lies at the origin. This is equivalent to supposing that, for all j,

$$(1) \qquad |f_j| + |g_j| > 0.$$

Since multiplication of Φ by e^{ki} does not affect its zeros, *any*

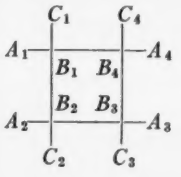
* Bôcher, Proceedings of the American Academy, vol. 40 (1904), pp. 469–484.

† The vector $F_j=f_j+g_j$ may be regarded as the force on a unit particle at z exerted according to the inverse distance law by a particle of mass α_j at a_j. See Bôcher, loc. cit., p. 475, footnote.

theorem true about the zeros of Φ for a given distribution D of the α_j's is likewise true for any distribution D' obtainable from D by a rotation of the α-plane about its origin. Our theorems will however be stated only for simple distributions of the α_j's, the understanding being, of course, that the same theorems hold for somewhat more general distributions.

3. *First Application.* The simplest cases with complex α_j's may be derived from the theorems of §1 by replacing points on the real axis by points on an arbitrary line through the origin. These cases we shall consider, however, by reason of the closing remark of §2, as already accounted for.

Let us assume at first instead that all the points α_j lie in* the first quadrant of the α-plane and the corresponding a_j in* the rectangle $B_1B_2B_3B_4$.



If z is any point in the angle $A_4B_4C_4$, the components of f_k and g_k parallel to A_1A_4 and C_1C_2 will be, respectively,

$$H_k = |f_k| \cos\phi_k + |g_k| \sin\phi_k,$$
$$V_k = |f_k| \sin\phi_k - |g_k| \cos\phi_k,$$

ϕ_k being the angle between f_k and a parallel to A_1A_4.

In order to locate the zeros of Φ in the angle $A_4B_4C_4$, we shall try to find points at which $\sum_1^n H_k = \sum_1^n V_k = 0$.

As $0 \leqq \phi_k \leqq \pi/2$, $H_k \geqq 0$. Because of (1), $H_k = 0$ for all k only in the following cases:

I. $\qquad f_k = \phi_k = 0, \qquad$ all k ;

II. $\qquad g_k = \phi_k - \dfrac{\pi}{2} = 0, \quad$ all k ;

III. $f_k = \phi_k = 0$, some k, $g_k = \phi_k - \pi/2 = 0$, remaining k.

In cases I and II, no zeros of Φ turn up because the V_k are of the same sign and therefore $\sum_1^n V_k \neq 0$.

In case III, on the contrary, where some of the a_j lie on

* In §§3, 4, 5, we shall use all our locative phrases "in," "to the right of," etc. in the wide sense, including thereby the boundary points of the regions considered. In §§6 and 7, we shall revert to the strict sense of these terms.

the line B_1B_4 and the corresponding α_j on the imaginary axis, and the remaining a_j lie on the line B_2B_4 and the corresponding α_j on the real axis, there is a possibility of $\sum_1^n V_k = 0$ and of a zero of Φ at the point B_4.

Similar considerations hold in the remaining angles $A_kB_kC_k$. *There are no zeros of Φ in any of the angles $A_kB_kC_k$, except perhaps at just one of the points B_k. If, furthermore, all the a_k lie on B_1B_4, there are no zeros to the right of C_3C_4 or to the left of C_1C_2.*

4. *Corollaries.* More elegant results for the case that the α_j's are in the first quadrant can now be obtained if the corresponding α_j are required to lie in a circle C of radius R. In the concentric circle C' of radius $2^{1/2}R$, a square S can be inscribed which will circumscribe C and which, when suitably chosen, will include in an angle $A_kB_kC_k$ any point outside C'. Consequently *there are no zeros of Φ outside of C'.*

If $n=2$, "outside of C'" must be taken in the strict sense. If a_1 and a_2 are situated on C, $\pi R/2$ units apart, S may have a_1 on one side and a_2 on the adjacent side. Case III of §3 would then permit a zero to be on C', as the example

$$\frac{i}{z-i}+\frac{1}{z-1}=0$$

with the unit circle as C, shows.

If the points a_j just considered are all on the line-segment AB, Φ will have a larger zero-free region. On AB as diagonal a rectangle R can so be constructed as to include in an angle $A_kB_kC_k$ any point outside of the circle Γ having AB as diameter. *Outside of Γ there are therefore no zeros of Φ.*

Again "outside of Γ" must be given a strict meaning when $n=2$. If a_1 and a_2 are at A and B, case III arises with its possibility of a zero on Γ, as for instance in the example

$$\frac{1}{z-1}+\frac{i}{z+1}=0,$$

where A: $(-1, 0)$ and B: $(1, 0)$.

5. *Other Applications.* By essentially the methods of §3, the following results may also be deduced:

(a) *Suppose the α_j to consist of two groups G_1 and G_2, the former situated in the first quadrant and the latter in the second quadrant of the α-plane. If the a_j corresponding to G_1 and G_2 lie respectively in the angles $A_1B_1C_2$ and $A_4B_4C_3$, there are no zeros of Φ in the strip $C_1B_1B_4C_4$, save perhaps upon the segment B_1B_4.*

The exceptions may occur only if $G_1=G_{1a}+G_{1c}$ and $G_2=G_{2a}+G_{2c}$, where G_{1a} and G_{2a} have their α_j's on the imaginary axis, G_{1c} and G_{2c} their α_j's on the real axis, and G_{1a}, G_{1c}, G_{2a}, and G_{2c} have their a_j's respectively on the segments A_1B_1, B_1C_2, A_4B_4, B_4C_3.

If $G_{1c}=G_{2c}=0$ (that is, G_{1c} and G_{2c} contain no points), any point of B_1B_4 may be a zero of Φ.

If $G_{1a}\cdot G_{1c}\neq 0$ and $G_{2c}=0$, the point B_1 is the only possible zero of Φ on B_1B_4.

If $G_{1c}=0$ and $G_{2a}\cdot G_{2c}\neq 0$, the point B_4 is the only possible zero of Φ on B_1B_4.

If the lines C_1C_2 and C_3C_4 are coincident, we shall exclude the case $G_{1a}=G_{2a}=0$, but include the cases

$$G_{1a}=0 \text{ and } G_{2a}\cdot G_{2c}=0,$$

$$G_{2a}=0 \text{ and } G_{1a}\cdot G_{1c}=0,$$

and

$$G_{1a}\cdot G_{1c}\cdot G_{2a}\cdot C_{2c}\neq 0,$$

in which cases B_1 is the only possible zero of Φ on B_1B_4.

(b) *Suppose the α_j to consist of four groups $G_k(k=1, 2, 3, 4)$, the group G_k being in the kth quadrant of the α-plane. If the a_j corresponding to G_k lie in the angle $A_kB_kC_k$, there are no zeros of Φ in the rectangle $B_1B_2B_3B_4$.*

We are supposing no two sides of $B_1B_2B_3B_4$ to coincide. A degenerate case of interest is, however, one in which A_1A_4 and A_2A_3 are coincident and the a_j corresponding to G_k lie on A_1B_1 if $k=1$ or 2, and on A_4B_4 if $k=3$ or 4. Then, if not all the α_j's lie on the real axis, there are no zeros of Φ on the segment B_1B_4. The theorems of (a) and (b) also hold for $n=\infty$ provided Φ is absolutely convergent in the above-mentioned zero-free regions.

6. *A Jensen Case.* So far no restrictions have been placed upon the behavior of Φ for real values of z. In this section we shall assume, in order to make Φ real on the real axis, that the a_j and the corresponding α_j are both real or both in conjugate imaginary pairs. We shall also suppose the α_j corresponding to an a_j in the upper half-plane also to be in the upper half-plane, and vice-versa. Furthermore, we shall limit the α_j to the first and fourth quadrants of the α-plane.*

Let us construct in the z-plane the circles (the Jensen circles) having as diameters the line segments joining pairs of conjugate a_j, and the indefinite lines (the Jensen lines) through the pairs of conjugate a_j. If we denote by R the region consisting of all points simultaneously outside of all the Jensen circles and to the left of all the Jensen lines, we may define R by the inequalities

$$(x - b_j)^2 + y^2 - c_j^2 > 0, \quad x - b_j < 0 ;$$

all j corresponding to imaginary $a_j = b_j + ic_j$.

In the sum Φ let us discern two types of terms, one corresponding to real a_j:

$$t_1 = \frac{\beta_j}{z - b_j} = \frac{\beta_j[(x - b_j) - iy]}{(x - b_j)^2 + y^2}, \qquad (\alpha_j = \beta_j + \gamma_j),$$

and the other corresponding to a pair of conjugate a_j:

$$t_2 = \frac{\beta_j + i\gamma_j}{z - (b_j + ic_j)} + \frac{\beta_j - i\gamma_j}{z - (b_j - ic_j)} .$$

The latter has as component parallel to the y-axis:

$$I(t_2) = \frac{-2y[\beta_j\{(x - b_j)^2 + y^2 - c_j^2\} - 2c_j\gamma_j(x - b_j)]}{\{(x - b_j)^2 + (y - c_j)^2\}\{(x - b_j)^2 + (y + c_j)^2\}} .$$

As $\beta_j \geqq 0$, $c > 0$, and $\gamma > 0$, we find in R

$$\operatorname{sgn} I(t_1) = \operatorname{sgn} I(t_2) = -\operatorname{sgn} y.$$

The components of t_1 and t_2 parallel to the y-axis will there-

* See the footnote to §3.

fore not vanish at points in R off the real axis. That is to say, *there are no imaginary roots of* Φ *in* R.

The question of the real roots of Φ in R may be easily settled through a study of the real function Φ on the real axis. Again we shall separate the terms corresponding to real a_j from those corresponding to pairs of conjugate a_j. The first

$$S_1 = \frac{\beta_j}{x - b_j} \quad \text{has} \quad \frac{dS_1}{dx} = -\frac{\beta_j}{(x - b_j)^2};$$

the second

$$S_2 = \frac{\beta_j + i\gamma_j}{x - (b_j + ic_j)} + \frac{\beta_j - i\gamma_j}{x - (b_j - ic_j)},$$

$$\frac{dS_2}{dx} = \frac{2\beta_j\{c_j^2 - (x - b_j)^2\} + 4c_j\gamma_j(x - b_j)}{\{(x - b_j)^2 + c_j^2\}^2}.$$

Therefore $d\Phi/dx < 0$ in R. In addition, as x moves to the right from one a_j to the next a_j, Φ changes from $+\infty$ to $-\infty$. In other words, any interval of the real axis in R not containing any a_j has at most one root of Φ; between two successive a_j in R, there is exactly one root of Φ. In short, *if the number of* a_j *in* R *is* k, *the number of real roots of* Φ *in* R *is* k, $k-1$, *or* $k+1$.*

The above equations also allow us to deduce at once the results for the case in which the α_j corresponding to real a_j lie on the positive real axis, and the remaining α_j are in the second and third quadrants of the α-plane. The above theorems will hold for this case if R in these theorems is replaced by K, the region consisting of all points simultaneously inside of all the Jensen circles and to the left of all the Jensen lines. (K may not contain any points.)

These theorems will also be valid when $n = \infty$ provided Φ converges uniformly in any closed region in R or K not including any points a_j, and all the imaginary a_j lie to the right of a fixed line parallel to the y-axis.

* E. B. Van Vleck has an unpublished proof of Jensen's theorem which uses similar analysis.

7. *The Function* $\eta(z)$. As a final application particularly of the methods of §6, we turn to the function

$$\eta(z) = \frac{1}{z}\left\{\frac{\zeta'(z)}{\zeta(z)} - (1 + \log 2\pi)\right\}$$
$$= \frac{2}{z^2} - \frac{1}{z-1} - \sum_{k=1}^{\infty} \frac{1}{k(z+2k)} + \sum_{-\infty}^{\infty} \frac{1}{\rho_n(z-\rho_n)},$$

where $\zeta(z)$ is the Riemann zeta function, and the ρ_n its zeros.

Concerning the ρ_n, it is known* that

$$\rho_{-n} = \bar{\rho}_n$$
$$0 < R(\rho_n) < 1, \qquad \text{(all } n\text{)},$$
$$R(\rho_n) = \tfrac{1}{2}, \qquad \text{for } |I(\rho_n)| \leqq 300,$$

and that $\rho = \frac{1}{2} \pm Ki$, $K = 14.13472(5)$, are the nearest ρ_n to the real axis.

Let us draw a circle of radius K, center at $(1/2, 0)$ and denote by S the interior of this circle to the left of the imaginary axis. Then in S there are no imaginary zeros of $\eta(z)$. In each of the intervals $(0, -2)$, $(-2, -4)$, $(-4, -6)$, $(-6, -8)$, $(-8, -10)$, and $(-10, -12)$ of the real axis there is exactly one zero, and in the interval $(-12, \frac{1}{2} - K)$ there is at most one zero of $\eta(z)$.

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* See Hutchinson, Transactions of this Society, vol. 27 (1925), p. 49, and references therein. My attention was called to the logarithmic derivative of the zeta function by C. E. Hille.

THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS IN A SPACE OF $(n+1)$ DIMENSIONS*

BY D. R. DAVIS

1. *Introduction.* Among the various types of inverse problems of the calculus of variations are those of Darboux, Hamel, Hirsch, and Kürschak.† Darboux discussed the problem of the plane showing that for a given equation of the form $y''=\phi(x, y, y')$ there exist an infinity of functions $f(x, y, y')$ such that the integral $\int_{x_1}^{x_2} f(x, y, y')dx$ taken along one of the integral curves of the given equations furnishes a maximum or minimum. Hamel found the general type of integral whose minimizing arcs are straight lines. Of the last two Hirsch considers an equation of the type $F(x, y, y', y'', \cdots, y^{(n)})=0$ and Kürschak generalizes this by introducing n independent variables. In both of these cases it was found that a necessary and sufficient condition for a given equation of the type considered to give a solution of a problem in the calculus of variations is that it have its equation of variation self-adjoint. No such restriction was found in Darboux's problem; however, it is well known that every differential equation of the second order for plane curves may be transformed into one whose equation of variation is self-adjoint.

The inverse problem of the calculus of variations for three-dimensional space is treated in my thesis.‡ It is the

* Presented to the Society, San Francisco Section, June 2, 1928.

† Darboux, *Théorie des Surfaces*, vol. 3, §606.

A. Hirsch, *Ueber eine Charakteristische Eigenshaft der Differentialgleichungen der Variationsrechnung*, Mathematische Annalen, vol. 49, p. 49.

J. Kürschak, *Ueber die Transformation der partiellen Differentialgleichungen der Variationsrechnung*, Mathematische Annalen, vol. 56, p. 155.

G. Hamel, *Geometrieen, in denen die Geraden die Kurzesten sind*, Mathematische Annalen, vol. 59, p. 255.

‡ *Inverse problem of the calculus of variations in higher space*, written under the direction of G. A. Bliss, University of Chicago, 1926. Published in the Transactions of this Society, vol. 30 (1928), pp. 710–736.

purpose of this paper to discuss the corresponding problem for a space of $(n+1)$ dimensions.

2. *Fundamental Properties of given Differential Equations.* Let us consider a system of n differential equations of the form

$$H_j(x, y_i, y_i', y_i'') = 0, \quad (i,j = 1, \cdots, n), \tag{1}$$

whose solutions are

$$y_i = y_i(x);$$

these have the derivatives y_i', y_i'' with respect to x. Under the hypothesis that the equations of variation of the given equations (1) form a self-adjoint system,* a function $f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ can be determined such that the given equations are the differential equations for the solutions of the problem of minimizing the integral

$$I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx. \tag{2}$$

The self-adjoint conditions which are needed here are summarized in the following theorem which is fully treated in the reference cited below.†

THEOREM. *Necessary and sufficient conditions that the system of differential expressions*

$$J_i(u) \equiv A_{ik}(x)u_k + B_{ik}(x)u_k' + C_{ik}(x)u_k'', \quad (i,k = 1, 2, \cdots, n),$$

shall be self-adjoint are

$$C_{ik} \equiv C_{ki},$$
$$B_{ik} + B_{ki} \equiv 2C_{ik}',$$
$$A_{ik} \equiv A_{ki} - B_{ki}' + C_{ki}''.$$

In the above and following expressions, the notation of

* This condition is also necessary; see Theorem II of my thesis, loc. cit., or J. Hadamard, *Leçons sur le Calcul des Variations*, p. 156.

† See my thesis, loc. cit.

tensor analysis is used, that is, whenever two subscripts are alike in two factors of a term, say of the form $A_{ik}u_k$, then the expression represents a sum with respect to the repeated index.

The equations of variation of the system (1) are

$$H_{iy_j''}u_j'' + H_{iy_j'}u_j' + H_{iy_j}u_j = 0. \tag{3}$$

For this system the self-adjoint conditions of the above theorem give respectively the following relations:

$$\begin{aligned} H_{iy_j''} &= H_{jy_i''}, \\ H_{jy_i'} + H_{iy_j'} &= 2(H_{iy_j''})', \\ H_{jy_i} &= H_{iy_j} - (H_{iy_j'})' + (H_{iy_j''})'', \end{aligned} \tag{4}$$

which must be identities in x, y_i', y_i', y_i'''.

The second set of relations (4) assert that each of the functions $H_i (i=1, \cdots, n)$ is linear in y_k'' $(k=1, \cdots, n)$, since terms in y_k''' do not occur in the first members. Therefore, the given functions (1) may be written in the form

$$\begin{aligned} H_i = M_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \\ + P_{ij}(x, y_1, \cdots, y_n, y_1', \cdots, y_n')y_j''. \end{aligned} \tag{5}$$

In this notation the first of relations (4) becomes

$$P_{ij} = P_{ji}. \tag{6}$$

From the last two of relations (4) we have

$$H_{jy_i} - H_{iy_j} = [(H_{iy_j''})' - H_{iy_j'}]' = \tfrac{1}{2}(H_{jy_i'} - H_{iy_j'})'. \tag{7}$$

Since the coefficient of y_k''' in the expansion of the second member of this equation must vanish we have the following conditions:

$$H_{jy_i'y_k''} = H_{iy_j'y_k''}. \tag{8}$$

In the notation of (5) these relations become

$$P_{jky_i'} = P_{iky_j'}. \tag{9}$$

From these conditions and (6) it follows that the expression

$P_{jky_i'}$ remains unchanged under all permutations of the indices i, j, k.

From (4_2) with the aid of (5) and (9), we obtain

$$
\begin{aligned}
M_{iy_j'} + M_{jy_i'} &= 2P_{ij}' - P_{iky_j'}y_k'' - P_{jky_i'}y_k'' \\
&= 2(P_{ijx} + P_{ijy_k}y_k' + P_{ijy_k'}y_k'') \\
&\quad - P_{jky_j'}y_k'' - P_{jky_i'}y_k'' \\
&= 2(P_{ijx} + P_{ijy_k}y_k').
\end{aligned} \tag{10}
$$

By applying relations (7) to the expressions for H_i given in (5) and with the use of (9), we get

$$
M_{jy_i} - M_{iy_j} = (P_{iky_j} - P_{jky_i})y_k'' + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})'. \tag{11}
$$

If these equations are to be identities in the variables involved, we must have

$$
\left\{
\begin{aligned}
&P_{iky_j} - P_{jky_i} = \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'})_{y_k'}, \\
&M_{jy_i} - M_{iy_j} = \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_x \\
&\qquad\qquad + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_{y_k}y_k'.
\end{aligned}
\right. \tag{11_a}
$$

The first of these systems may be obtained from (10) with the use of (9) but the second is an independent system of relations which must be satisfied.

The above results may be summarized as follows:

THEOREM 1. *If a system of differential equations of the second order*

$$
H_j(x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0, \qquad (j = 1, \cdots, n),
$$

is to have equations of variation which are self-adjoint along every curve $y_i = y_i(x)$, then it must have the form

$$
\begin{aligned}
H_i = {} & M_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \\
& + P_{ij}(x, y_1, \cdots, y_n, y_1', \cdots, y_n')y_j'',
\end{aligned}
$$

$(i, j = 1, \cdots, n)$ where the functions M_i and P_{ij} satisfy the conditions

$$(12)\qquad \begin{cases} P_{ij} = P_{ji}\,; \quad P_{jky_{i'}} = P_{iky_{j'}}, \\ M_{jy_{i'}} + M_{iy_{j'}} = 2(P_{ijx} + P_{ijy_k}y_k'), \\ M_{jy_i} - M_{iy_j} = \tfrac{1}{2}(M_{jy_{i'}} - M_{iy_{j'}})_x \\ \qquad\qquad + \tfrac{1}{2}(M_{jy_{i'}} - M_{iy_{j'}})_{y_k}y_k', \end{cases}$$

identically in x, y_i, y_i'.

3. *Determination of the Integrand f of the Integral I.* For a system of equations $H_j = 0$ which has the form indicated in the above theorem with coefficients satisfying (12), it is possible to determine the integrand f of the integral I in (2) to an arbitrary function of $x, y_1, \cdots, y_n$ such that $H_j = 0$ are the equations of its minimizing curves.

A function $g(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ will first be determined satisfying the relations

$$(13)\qquad g_{y_{i'}y_{j'}} = P_{ij}, \qquad (i,j = 1, \cdots, n).$$

The relations (12_2) are necessary and sufficient conditions for the integrability of system (13). The function g will be determined by the integral

$$g = \int_{y_{10},\cdots,y_{n0}}^{y_1',\cdots,y_n'} L_1dy_1' + L_2dy_2' + \cdots + L_ndy_n',$$

where

$$L_1 = \int_{y'_{10},\cdots,y_{n'0}}^{y_1',\cdots,y_n'} P_{11}dy_1' + P_{12}dy_2' + \cdots + P_{1n}dy_n',$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$L_n = \int_{y'_{10},\cdots,y'_{n0}}^{y'_1,\cdots,y_n'} P_{n1}dy_1' + P_{n2}dy_2' + \cdots + P_{nn}dy_n'.$$

If g is a particular solution of the system (13), the most general solution is given by the formula

$$(14)\quad f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + a(x, y_1, \cdots, y_n) + b_k(x, y_1, \cdots, y_n)y_k', \quad (k = 1, \cdots, n),$$

where $a, b_1, \cdots, b_n$ are arbitrary functions of $x, y_1, \cdots, y_n$.

If the given equations are identical with the Euler-Lagrange equations for the integral (2) then the equations

$$\frac{d}{dx}(f_{y_i'}) - f_{y_i} = M_i + P_{ij}y_j'' \tag{15}$$

must be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n''$. The value of f given in (14) must now satisfy the system (15); thus we have

$$\frac{d}{dx}(g_{y_i'} + b_i) - (g_{y_i} + a_{y_i} + b_{ky_i}y_k') = M_i + P_{ik}y_k'',$$

which readily reduce to

$$(b_{ix} - a_{y_i}) + (b_{iy_k} - b_{ky_i})y_k' = M_i + g_{y_i} - [g_{y_i'x} - g_{y_i'y_k}y_k']. \tag{16}$$

If these equations are to be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ the expressions on the right must be independent of y_i' and linear in y_k' when $k \neq i$. Upon differentiating the second members of (16) with respect to y_i' one obtains

$$M_{iy_i'} - g_{y_i'y_i'x} - g_{y_i'y_i'y_k}y_k',$$

which by conditions (12_3), for $i=j$, are identically zero. Differentiation with respect to y_j' $(j \neq i)$, gives

$$b_{iy_j} - b_{jy_i} = M_{iy_j'} + g_{y_iy_j'} - g_{y_i'y_j} - g_{y_i'y_j'x} - g_{y_i'y_j'y_k}y_k'.$$

With the aid of (13) and (12_3) this system reduces to

$$b_{iy_j} - b_{jy_i} = \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'}) + g_{y_iy_j'} - g_{y_i'y_j}. \tag{17}$$

If in these equations we again take the partial derivatives, with respect to y_j', the functions on the right yield the following expressions:

$$\tfrac{1}{2}(M_{iy_j'} - M_{jy_i'})_{y_j'} + P_{ji}y_i - P_{ij}y_j. \tag{18}$$

By setting $k=j$ in the first system of (11_a) we see that the expressions (18) are each identically zero. Thus we have shown that the second members of (16) are, as indicated

by the expressions on the left, independent of y_i and linear in y_k for $k \neq i$.

Upon substituting the values of the first members of (17) in (16) we see that the system (16) which are partial differential equations for a, $b_1, \cdots, b_n$ may be replaced by the following:

$$(19)\qquad \begin{aligned} b_{iy_i} - b_{iy_j} &= g_{y_i'y_j} - g_{y_iy_j'} - \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'}), \\ b_{jx} - a_{y_j} &= M_j + g_{y_jx} - g_{y_j'x} - g_{y_jy_k'}y_k' \\ &\quad - \tfrac{1}{2}(M_{jy_k'} - M_{ky_j'})y_k'. \end{aligned}$$

In (19_1) there are $n(n-1)/2$ equations and in (19_2) n equations in the $n+1$ variables x, $y_1, \cdots, y_n$. The determinant of the matrix of the functional expressions on the right in (19_1) is skew symmetric. Solutions for the system (19) exist if the totality of $n(n+1)/2$ equations are compatible.

We shall now prove the following theorem.

THEOREM 2. *A necessary and sufficient condition that there exist a solution for a system of differential equations of the form*

$$(20)\qquad b_{iy_i} - b_{iy_j} = \phi_{ij}, \quad b_{jx} - a_{y_j} = \theta_j,$$

where a, ϕ_{ij}, b_j, θ_j, $(i, j = 1, 2, \cdots, n)$, *are functions of* $x, y_1, \cdots, y_n$ *and* $\phi_{ij} = -\phi_{ji}$, *is that*

$$(21)\qquad \phi_{ijx} - \theta_{iy_i} + \theta_{iy_j} \equiv 0$$

identically in $x, y_1, \cdots, y_n$ *for every pair of values of* i *and* j.

That the condition is necessary is evident. In order to prove that it is also sufficient consider the given equations (20) for fixed values of i and j, namely, $i = p$, $j = q$. For these values of i and j system (20) gives

$$(22)\qquad b_{qy_p} - b_{py_q} = \phi_{pq}, \quad a_{y_q} - b_{qx} = -\theta_q, \quad b_{px} - a_{y_p} = \theta_p.$$

Now since

$$\phi_{pqx} - \theta_{qy_p} + \theta_{py_q} \equiv 0,$$

identically in x, y_p, y_q, there exists a solution, say a,

b_p, b_q, of equations (22)*. Now let $j=r$, where r is any other value of j than q. The substitution of the functions a, b_p, b_r in the equations for $i=p$, $j=r$ gives

$$(23)\quad b_{ry_p} - b_{py_r} = \phi_{pr}, \quad a_{y_r} - b_{rz} = -\theta_r, \quad b_{pz} - a_{y_p} = \theta_p.$$

Since the last of these equations is identical with the last equation of (22), it is satisfied by the functions a, b_p. Solving for the derivatives of b_r in the first two equations, we obtain

$$b_{ry_p} = \phi_{pr} + b_{py_r}, \quad b_{rz} = \theta_r + a_{y_r},$$

which are compatible provided that

$$\phi_{prz} + b_{py_rz} = \theta_{ry_p} + a_{y_ry_p}.$$

By employing the last of equations (22) this equation becomes

$$\phi_{prz} - \theta_{ry_p} + \theta_{py_r} = 0,$$

which is identically zero by hypothesis. A similar condition prevails for $i=s$, $j=q$, where s is any other value of i than p. It follows that the remaining $(n+3)(n-2)/2$ equations are compatible for a solution of the given equations for $i=p$, $j=q$. Hence, there exists a solution of the given system.

System (20) is identical with (19) if ϕ_{ij} and θ_j represent the second members of (19) respectively. Relations (21) applied to the second members of (19) give the following system:

$$(24)\quad 0 = (M_{iy_j} - M_{jy_i}) - \tfrac{1}{2}(M_{iy_j'} - M_{jy_i'})_z + \tfrac{1}{2}(M_{jy_k'} - M_{ky_j'})_{y_i}y_k' + \tfrac{1}{2}(M_{ky_i'} - M_{iy_k'})_{y_j}y_k'.$$

These equations are identities in x, $y_1, \cdots, y_n$, $y_1', \cdots, y_n'$, provided the following system is identically zero:

$$(25)\quad (M_{jy_k'} - M_{ky_j'})_{y_i}y_k' + (M_{ky_i'} - M_{iy_k'})_{y_j}y_k' - (M_{iy_i'} - M_{iy_j'})_{y_k}y_k' \equiv 0,$$

* See H. Weber, *Die Partiellen Differentialgleichungen*, vol. 1, pp. 221, etc.

which is obtained by replacing the first two expressions in parenthesis of (24) by their value obtained from the last system of (12).

It is readily verified that those terms of (25) for which any two of the indices i, j, k are equal vanish identically. It remains to show that the terms for which $k \neq i \neq j$ will also vanish. For this purpose consider the last of the self-adjoint relations (12). Let this system be written successively for $j=p, i=r; j=q, i=p; j=r, i=q$. If these three equations are differentiated with respect to y_q', y_r', y_p' respectively and the results added, we obtain equations (25) except for the factor y_k'. Hence, equations (25) are identically zero in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$.

Since the conditions of Theorem 2 are satisfied there exists a solution of the system (19). Let a, b_i, b_j be a particular solution of system (19); then the most general solution will be

$$
\begin{aligned}
&a(x,y_1,\cdots,y_n)+u(x,y_1,\cdots,y_n),\\
&b_i(x,y_1,\cdots,y_n)+v_i(x,y_1,\cdots,y_n),\\
&b_j(x,y_1,\cdots,y_n)+v_j(x,y_1,\cdots,y_n), \qquad (i<j),
\end{aligned} \tag{26}
$$

where the functions u, v_i, v_j satisfy the relations

$$
v_{jy_i}-v_{iy_j}=0, \quad u_{y_j}-v_{jx}=0, \quad v_{ix}-u_{y_i}=0, \quad (i>j). \tag{27}
$$

The general solution of the system (19) may now be written in the form

$$
\begin{aligned}
&a(x,y_1,\cdots,y_n)+u(x,y_1,\cdots,y_n),\\
&b_k(x,y_1,\cdots,y_n)+v_k(x,y_1,\cdots,y_n),\\
&\qquad\qquad (k=1,\cdots,n).
\end{aligned} \tag{28}
$$

Relations (27) are necessary and sufficient conditions that the expression

$$
u+v_ky_k', \qquad (k=1,\cdots,n),
$$

should be the total derivative of an arbitrary function

$t(x, y_1, \cdots, y_n)$. In view of (28), the integrand f of (14) takes the form

$$(29)\quad f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + a(x, y_1, \cdots, y_n) + b_k(x, y_1, \cdots, y_n)y_k' + \frac{d}{dx}t(x, y_1, \cdots, y_n),$$

where $a, b_1, \cdots, b_n$ are solutions of the system (19) and t is an arbitrary function of $x, y_1, \cdots, y_n$.

THEOREM 3. *If a system of differential equations of the form*

$$H_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0, \qquad (i = 1, \cdots, n),$$

has equations of variation

$$H_{iy_j''}u_j'' + H_{iy_j'}u_j' + H_{iy_j}u_j = 0,$$

which form a self-adjoint system along every curve $y_i = y_i(x)$, then there exists an integral of the form

$$I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

having the equations $H_i(x, y_j, y_j', y_j'') = 0$ as its Euler equations. The most general such integral has an integrand

$$f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + a(x, y_1, \cdots, y_n) + b_k(x, y_1, \cdots, y_n)y_k' + \frac{d}{dx}t(x, y_1, \cdots, y_n)$$

where g is a particular solution of (13), *$a, b_1, \cdots, b_n$ are a particular solution of system* (19), *and t is an arbitrary function of $x, y_1, \cdots, y_n$.*

THE UNIVERSITY OF OREGON

ON THE MAPPING OF THE QUADRUPLES OF THE INVOLUTORIAL G_4 IN A PLANE UPON A STEINER SURFACE*

BY ARNOLD EMCH

1. *Introduction.* Castelnuovo has definitely shown† that every plane involution may be mapped uniformly upon a rational surface. As may be expected, and as the author has shown in case of the involution of sextuples,‡ from such a mapping process arise interesting properties of certain configurations and curves which reflect geometric properties of the corresponding surface, and conversely. In this paper the involution induced by the group

$$G_4 \equiv \begin{pmatrix} \pm x_1, & \pm x_2, & \pm x_3 \\ x_1, & x_2, & x_3 \end{pmatrix}$$

is investigated from this standpoint.

In what follows I shall denote the involutorial quadruple in the plane (x) merely by G_4. To construct this, let $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$ be the coordinate triangle, and $B(x_1, x_2, x_3)$ a generic point. Join B to A_1, A_2, A_3 and construct the fourth harmonic lines to BA_1, BA_2, BA_3 with respect to the pairs of sides A_1A_2, A_1A_3; A_2A_3, A_2A_1; A_3A_1, A_3A_2. These three lines intersect in the triangle $B_1(-x_1, x_2, x_3)$, $B_2(x_1, -x_2, x_3)$, $B_3(x_1, x_2, -x_3)$, which together with $B(x_1, x_2, x_3)$ form the G_4. If we construct for each B_i the symmetric G_6 and denote it by (B_i), we obtain the configuration of the octahedral group

$$G_{24} \equiv \begin{pmatrix} \pm x_i, & \pm x_k, & \pm x_l \\ x_1, & x_2, & x_3 \end{pmatrix},$$

* Presented to the Society, February 23, 1929.

† *Sulla razionalità delle involuzioni piane*, Mathematische Annalen, vol. 44 (1894), pp. 125–155.

‡ *On the mapping of the sextuples of the symmetric substitution group G_6 in a plane upon a quadric*, this Bulletin, vol. 33 (1927), pp. 745–750.

consisting of the four sextuples (B), (B_1), (B_2), (B_3), which lie each on a conic as is well known.*

2. *Mapping on the Steiner Surface.* The substitutions of the G_4 outside of identity, expressed as collineations, are the three involutorial perspectives

$$I_1 \equiv (\rho x_1' = -x_1,\ \rho x_2' = x_2,\ \rho x_3' = x_3),$$
$$I_2 \equiv (\rho x_1' = x_1,\ \rho x_2' = -x_2,\ \rho x_3' = x_3),$$
$$I_3 \equiv (\rho x_1' = x_1,\ \rho x_2' = x_2,\ \rho x_3' = -x_3),$$

which leave the quadruple G_4 invariant. If we propose to construct those integral rational homogeneous functions in (x) which are invariant under these perspectives, we must form homogeneous functions of the squares of the variables: $F(x_1^2, x_2^2, x_3^2)$. The simplest set of these is evidently $a_1x_1^2+a_2x_2^2+a_3x_3^2=0$, of which type there are only three linearly independent. These conics form a net with $A_1A_2A_3$ as a common self-polar triangle. Next in order is the quartic

$$(1)\quad b_1x_1^4 + b_2x_2^4 + b_3x_3^4 + c_1x_2^2x_3^2 + c_2x_3^2x_1^2 + c_3x_1^2x_2^2 = 0,$$

which may be reduced by a collineation to the form

$$(2)\quad a_1x_2^2x_3^2 + a_2x_3^2x_1^2 + a_3x_1^2x_2^2 + a_4(x_1^4 + x_2^4 + x_3^4) = 0,$$

so that all quartics of the webb (2) may be expressed linearly by the relations

$$(3)\quad \rho y_1 = x_2^2x_3^2,\quad \rho y_2 = x_3^2x_1^2,\quad \rho y_3 = x_1^2x_2^2,$$
$$\rho y_4 = x_1^4 + x_2^4 + x_3^4.$$

From (3) it is evident that to the quadruple G_4 in (x) corresponds in the space (y) uniquely a point. The locus of these points is easily obtained as the Steiner surface Γ_4

$$(4)\quad y_1^2y_2^2 + y_2^2y_3^2 + y_3^2y_1^2 - y_1y_2y_3y_4 = 0.$$

Hence we may state the following theorem.

* Emch, A., *Some geometric applications of symmetric substitution groups*, American Journal of Mathematics, vol. 45 (1923), pp. 192–207.

THEOREM 1. *By the transformation* (3) *the Steiner surface* (4) *is mapped upon a plane, so that to a generic point on the surface correspond four points* B, B_1, B_2, B_3 *of a self-projective quadruple in the group of involutions* I, I_1, I_2, I_3.

3. *Properties of Quadruples.* To a plane section of Γ_4 by $a_1y_1+a_2y_2+a_3y_3+a_4y_4=0$ corresponds in (x) the quartic (2). In particular to a section of Γ_4 by a plane through the triple-point A_4 of Γ_4 corresponds the quartic

$$a_1x_2^2x_3^2 + a_2x_3^2x_1^2 + a_3x_1^2x_2^2 = 0, \tag{5}$$

and to the section by $y_4=0$ the quartic

$$x_1^4 + x_2^4 + x_3^4 = 0. \tag{6}$$

The quartic (5) is the transform of the conic

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0, \tag{7}$$

by the quadratic transformation $\rho x_1' = x_2x_3$, $\rho x_2' = x_3x_1$, $\rho x_3' = x_1x_2$. To the conic (7) corresponds on Γ_4 a conic which is the residual intersection by the quadric $a_1y_2y_3+a_2y_3y_1+a_3y_1y_2=0$. When a point B lies on (7), then all points of the quadruple G_4 determined by it lie on it. Now two generic points determine the conic (7), so that we have the following obvious theorem.

THEOREM 2. *Two arbitrary distinct quadruples* G_4 *lie on a definite conic of the net* (7).

A tangent plane at a generic point P of Γ_4 cuts the surface in two conics K_1 and K_2, whose four intersections lie at P and on A_4A_1, A_4A_2, A_4A_3. To these conics correspond in (x) two conics on the quadrangle corresponding to P. Obviously, any conic in (x) may be one of these conics. The product of the two conics in (x), corresponding to K_1 and K_2, must have the form (2). Hence when $a_1x_1^2+a_2x_2^2+a_3x_3^2=0$ is one of the conics, the other $b_1x_1^2+b_2x_2^2+b_3x_3^2=0$ must be such that their product will have the form (2). This is the case when $b_1=1/a_1$, $b_2=1/a_2$, $b_3=1/a_3$, so that their product,

$(a_1x_1^2+a_2x_2^2+a_3x_3^2)\,(a_2a_3x_1^2+a_3a_1x_2^2+a_1a_2x_3^2)=0$, when expanded in the form (2), becomes

$$(8)\quad a_1(a_2^2+a_3^2)x_2^2x_3^2+a_2(a_3^2+a_1^2)x_3^2x_1^2 + a_3(a_1^2+a_2^2)x_1^2x_2^2+a_1a_2a_3(x_1^4+x_2^4+x_3^4)=0.$$

The plane to whose section with Γ_4 corresponds in (x) the degenerate quartic (8) is therefore

$$(9)\quad a_1(a_2^2+a_3^2)y_1+a_2(a_3^2+a_1^2)y_2 + a_3(a_1^2+a_2^2)y_3+a_1a_2a_3y_4=0.$$

As the coordinates of this plane are $\rho u_1=a_1(a_2^2+a_3^2)$, $\rho u_2=a_2(a_3^2+a_1^2)$, $\rho u_3=a_3(a_1^2+a_2^2)$, $\rho u_4=a_1a_2a_3$, the envelope of the tangent-planes of the Steiner surface, by elimination of the a's, is easily obtained as

$$(10)\qquad 4u_4^3-(u_1^2+u_2^2+u_3^2)u_4+u_1u_2u_3=0,$$

a cubic of class 3, as is well known.

Consider now a generic line g in (y) cutting Γ in four points P_1, P_2, P_3, P_4. To these correspond in (x) four quadruples G_4^1, G_4^2, G_4^3, G_4^4, which together form the base points of the pencil of quartics corresponding to the plane sections through g. Through g we can draw three tangent planes to Γ_4 according to (10), which touch Γ_4 in T_1, T_2, T_3. To these T's correspond in (x) three quadruples D_1, D_2, D_3. The tangent planes t_1, t_2, t_3 cut Γ_4 in three pairs of conics, of which each passes through the four P's and one of the points T, and to which correspond in (x) also three pairs of conics. Each of the latter contains all four quadruples G_4^i, $i=1, 2, 3, 4$, and one of the quadruples D. Hence the quadruples G_4 and D are as entities precisely in the relation of a complete quadrangle with its three diagonal points.

The analogy of the quadruples G and D with a complete quadrangle leads to a very simple geometric proof that the Steiner surface is of class 3. Through the quadruples G we can pass three pairs of conics $K(G_4^1, G_4^2)$, $K(G_4^3, G_4^4)$; $K(G_4^1, G_4^3)$, $K(G_4^2, G_4^4)$; $K(G_4^1, G_4^4)$, $K(G_4^2, G_4^3)$, such that each pair contains all four quadruples. Each of these pairs

has a quadruple D as base points, and to each corresponds on Γ_4 two conics through P_1, P_2, P_3, P_4 and one of the points T. The plane of these conics touches Γ_4 at T. Hence there are three such tangent-planes through g to the Steiner surface. These results may be stated in the following form.

THEOREM 3. *The quadruples of the involutorial G_4 in the plane (x) may be arranged in groups of seven according to a complete quadrangle array. Any two quadruples in (x) determine such a configuration. Thus there are ∞^4 such configurations. To each such configuration correspond on the Steiner surface the four intersections of a line g with Γ_4 and the three points of tangency of the tangent-planes from g to Γ_4.*

4. *Quadruples on Quartics.* To a point on Γ_4:

$$y_1,\ y_2,\ y_3,\ (y_2^2 y_3^2 + y_3^2 y_1^2 + y_1^2 y_2^2)/(y_1 y_2 y_3),$$

corresponds in (x) the quadruple

$$[(y_2y_3)^{1/2},\quad (y_3y_1)^{1/2}, (y_1y_2)^{1/2}]\ ;\ [-(y_2y_3)^{1/2}, (y_3y_1)^{1/2},\quad (y_1y_2)^{1/2}]\ ;$$
$$[(y_2y_3)^{1/2}, -(y_3y_1)^{1/2}, (y_1y_2)^{1/2}]\ ;\ [\quad (y_2y_3)^{1/2}, (y_3y_1)^{1/2}, -(y_1y_2)^{1/2}].$$

To a point on $x_1=0$ corresponds on Γ_4, $\rho y_1 = x_2^2 x_3^2$, 0, 0, $\rho y_4 = x_2^4 + x_3^4$, that is, a definite point on A_4A_1 in (y). To this same point, corresponds in (x) a quadruple which lies on $x_1=0$, and which, from the form of the equation $x_2^4 - (y_1/y_4)x_2^2x_3^2 + x_3^4 = 0$, is easily seen to have the form $(0, x_2, x_3)$; $(0, x_2, -x_3)$; $(0, x_3, x_2)$; $(0, x_3, -x_2)$. The third and fourth of these points are obtained by projecting the inverse of (x_1, x_2, x_3) and $(x_1, x_2, -x_3)$ in the quadratic involution $x_i' = 1/x_i$ from A_1 in (x) upon $x_1=0$. The first two points, each counted twice, must be considered as a degenerate involutorial quadrangle. The same is true of the other two.

Now a generic line l in (x) cuts a quartic C_4 in four points; consequently to l corresponds on Γ_4 a space quartic L_4, since it cuts the curve D_4, corresponding to C_4, and its plane in four points. As l cuts each of the sides of the coordinate triangle in one point, L_4 cuts each A_4A_1, A_4A_2, A_4A_3 in (y), say in B_1, B_2, B_3. To L_4 correspond in (x) besides l the three

associated lines l_1, l_2, l_3 in the involution, which together form an involutorial quadrilateral L whose vertices lie two by two on the sides of the coordinate triangle. If $a_1x_1+a_2x_2+a_3x_3=0$ is the line l, then the pole of l with respect to the coordinate triangle $A_1A_2A_3$ is $E(a_1a_2,\ a_3a_1,\ a_1a_2)$ and its quadratic transform $F(a_1,\ a_2,\ a_3)$. The triangular polar of F is $m=a_2a_3x_1+a_3a_1x_2+a_1a_2x_3=0$. This determines another involutorial quadrilateral $M=mm_1m_2m_3$ in (x) whose vertices lie again by pairs on the sides of $A_1A_2A_3$. To these vertices correspond on Γ_4 the same points B_1, B_2, B_3 as in the case of L. To the quadrilateral M corresponds on Γ_4 a space quartic M_4 through $B_1B_2B_3$. The plane β of these points cuts Γ_4 in a quartic B_4, and L_4 and M_4 each cut B_4 in a fourth point outside of B_1, B_2, B_3. The plane β has the form

$$\beta = a_1^2(a_2^4+a_3^4)y_1+a_2^2(a_3^4+a_1^4)y_2 + a_3^2(a_1^4+a_2^4)y_3-a_1^2a_2^2a_3^2y_4=0. \tag{11}$$

Denoting the coordinates of this plane by u_1, u_2, u_3, u_4, its envelope is easily found as the class-cubic

$$4u_4^3-(u_1^2+u_2^2+u_3^2)u_4-u_1u_2u_3=0. \tag{12}$$

This is, however, different from the Steiner class-cubic. The plane β cuts Γ_4 in a quartic B_4 to which corresponds in (x) the quartic N:

$$a_1^2(a_2^4+a_3^4)x_2^2x_3^2+a_2^2(a_3^4+a_1^4)x_3^2x_1^2 + a_3^2(a_1^4+a_2^4)x_1^2x_2^2-a_1^2a_2^2a_3^2(x_1^4+x_2^4+x_3^4)=0. \tag{13}$$

N passes through the 12 vertices of the quadrilaterals L and M in (x), and moreover through the involutorial quadrangles corresponding to the fourth intersections of L_4 and M_4 with B_4. These results may be stated as follows.

THEOREM 4. *There exists in (x) a system of quartics which cut the sides of the fundamental triangle of the involution in 12 points of two complete quadrilaterals. To these correspond in (y) two space quartics through the same three points $B_1B_2B_3$*

on the double lines of Γ_4. *The locus of the planes* β *is a certain class-cubic.*

To the general invariant quartic of the involution

(14) $b_1x_1^4 + b_2x_2^4 + b_3x_3^4 + c_1x_2^2x_3^2 + c_2x_3^2x_1^2 + c_3x_1^2x_2^2 = 0$

corresponds on Γ_4 a space quartic cut out by the cone

(15) $b_1y_2^2y_3^2 + b_2y_3^2y_1^2 + b_3y_1^2y_2^2 + y_1y_2y_3(c_1y_1 + c_2y_2 + c_3y_3) = 0$,

which has A_4A_1, A_4A_2, A_4A_3 in (y) as nodal lines. When $b_1 = b_2 = b_3 = c_4$, then by means of Γ_4, (15) reduces to $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$, which verifies the results in case of a plane quartic on Γ_4.

Concerning the base points of a pencil of quartics of the general involutorial type we may state the following fact.

THEOREM 5. *The* 16 *base points of a pencil of involutorial quartics in* (x) *form* 4 *involutorial quadruples which lie two by two on* 6 *involutorial conics.*

5. *Flexes and Double Tangents.* If we draw the three tangent-planes from a line g to Γ_4, they cut Γ_4 in three pairs of conics through (P_1P_2, P_3P_4); (P_1P_3, P_2P_4); (P_1P_4, P_2P_3). With every couple P_iP_k is associated just one such conic, just as in (x) there is just one conic through the quadruples G_4^i and G_4^k. As two of the points P determine the remaining two on the line joining them, it is evident that *there is just one such conic through two generic points* P_i *and* P_k *on* Γ_4.

A generic plane cuts Γ_4 in a rational quartic to which corresponds in (x) a quartic C_4 without double points. This is clear, since to a double point of L on A_4A_1 in (y) correspond in (x) four distinct points on A_2A_3, and to every other point of L corresponds a proper involutorial quadruple on C_4. The Hessian of C_4 is a sextic H_6 which as a covariant of C_4 cuts C_4 in 6 involutorial flex-quadruples. Hence to H_6 corresponds on Γ_4 a sextic which cuts the quartic L, corresponding to C_4, in six points. These form 15 couples P_iP_k and through each of these couples passes a conic on Γ_4. Hence we may state the following theorem.

THEOREM 6. *The* 24 *flexes of a quartic* C_4 *in* (x) *lie on* 15 *conics, of which each contains* 8 *of the* 24 *flexes.*

This follows, of course, also from the fact that any two generic involutorial quadruples lie on a conic.

For a plane of the pencil $y_1+y_2+y_3+\lambda y_4=0$, the corresponding quartic C_4 in (x) becomes symmetric, so that with every point on C_4 are associated 5 other points which together lie on a conic. *Hence in this case, in addition to the* 15 *conics of the flexes there are* 4 *other conics of which each contains* 6 *of the flexes.*

The 28 double tangents of C_4 may be grouped in 7 involutorial quadrilaterals, each with two involutorial quadrangles of points of tangency, so that the 56 points of tangency form 14 involutorial quadrangles.

THEOREM 7. *The* 28 *double tangents of the* C_4 *form* 7 *involutorial quadrilaterals whose* 56 *points of tangency lie* 8 *by* 8 *on* $\binom{14}{2}=91$ *conics.*

6. *Tropes.* There are four tangent planes touching Γ_4 along conics. To these correspond in (x) four couples of coincident conics, which must have the form $(a_1x_1^2+a_2x_2^2+a_3x_3^2)^2 = a_1^2x_1^4+a_2^2x_2^4+a_3^2x_3^4+2(a_1a_2x_1^2x_2^2+a_2a_3x_2^2x_3^2+a_3a_1x_3^2x_1^2)=0$. But for a quartic corresponding to a plane section there must be $a_1^2=a_2^2=a_3^2$, which for (a_1, a_2, a_3) gives the possible solutions (a, a, a); $(-a, a, a)$; $(a, -a, a)$; $(a, a, -a)$. Consequently the four tropes of the Steiner surface are

$$
\begin{aligned}
2(y_1+y_2+y_3)+y_4&=0,\\
2(y_1-y_2-y_3)+y_4&=0,\\
2(-y_1+y_2-y_3)+y_4&=0,\\
2(-y_1-y_2+y_3)+y_4&=0.
\end{aligned}
\tag{16}
$$

7. *Quartics on* G_{48}. When (x) and (x') are connected by the quadratic involution $x_i'=1/x_i$, then the corresponding points $P(y)$ and $P'(y')$ on the map Γ_4 are also birationally connected. The corresponding involution on Γ_4 has the form

$$(17) \qquad \delta y_1' = y_2 y_3, \quad \delta y_2' = y_3 y_1, \quad \delta y_3' = y_1 y_2,$$
$$\delta y_4' = y_1'^2 + y_2'^2 + y_3'^2,$$

considering that $y_4 = (y_2^2 y_3^2 + y_3^2 y_1^2 + y_1^2 y_2^2)/(y_1 y_2 y_3)$. If we form the line coordinates $p_{ik} = y_i y_k' - y_k y_i'$ of PP', we obtain a rational congruence, whose intersection with a complex is a ruled surface which cuts Γ_4 in an invariant curve of the involution $(y) \rightleftarrows (y')$. To these correspond in (x) invariant curves of the involution $(x) \rightleftarrows (x')$. Such curves are for example the sextics

$$(18) \quad x_1^2(x_2^2 - x_3^2)^2 + x_2^2(x_3^2 - x_1^2)^2 + x_3^2(x_1^2 - x_2^2)^2 = 0,$$

which is the locus of couples of corresponding points (x), (x') on the tangents of the class-conic $u_1^2 + u_2^2 + u_3^2 = 0$, and

$$(19) \quad x_1^2(x_2^2 + x_3^2)^2 + x_2^2(x_3^2 + x_1^2)^2 + x_3^2(x_1^2 + x_2^2)^2 + \lambda x_1^2 x_2^2 x_3^2 = 0.$$

Both are invariant in the Cremona group G_{48}, which is the product of the quadratic involution and the octahedral group G_{24}.

The curve (18) has the vertices of the fundamental triangle and the four invariant points of the quadratic involution as double points and is of the genus three type.

8. *Clebsch and Lüroth Curves.* The quartics which are invariant under the octahedral group G_{24} have the form

$$(20) \qquad \Sigma x_i^4 + 6\lambda \Sigma x_i^2 x_k^2 = 0.$$

Some of these curves are rather well known.* The covariant S is in the pencil, and the parameter λ for S has the value $\lambda' = (\lambda^2 - 2\lambda^3 - \lambda^4)/(6\lambda^4)$. The question whether there are any Clebsch and Lüroth curves in the pencil can easily be answered by determining those values of λ for which the so-called Clebsch determinant of order 6 vanishes.

In case of the quartic (20), this determinant reduces to the form

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* Ciani, E., *I varî tipi possibili di quartiche piane piu volte omologico-armoniche*, Rendiconti di Palermo, vol. 13 (1899), pp. 347–373.

$$(21) \qquad \lambda^3 \begin{vmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{vmatrix} = 0.$$

The roots are $\lambda=0$, $\lambda=1$, $\lambda=-1/2$. The first two roots are three- and two-fold, respectively. For $\lambda=0$ we obtain the Dyck curve for which the covariant S becomes indeterminate. For $\lambda=1$ the apolar conic has the form $a(u_1^2-u_2^2)-b(u_2^2-u_3^2)=0$, and (20) becomes

$$C_4 = (x_1+x_2+x_3)^4+(-x_1+x_2+x_3)^4 + (x_1-x_2+x_3)^4+(x_1+x_2-x_3)^4=0,$$

$$S = (x_1+x_2+x_3)(-x_1+x_2+x_3) \cdot(x_1-x_2+x_3)(x_1+x_2-x_3)=0.$$

For $\lambda=-1/2$, the apolar conic simply assumes the form $u_1^2+u_2^2+u_3^2=0$, and

$$C_4 = x_1^4+x_2^4+x_3^4-3(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2^2)=0,$$

$$S = x_1^4+x_2^4+x_3^4+7(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2^2)=0.$$

These Clebsch and Lüroth curves invariant under the octahedral collineation group seem to be new.

The construction of these curves, outside its tediousness, does not present particular difficulties. Any tangent t_1 of the apolar conic cuts S in four points from which four other tangents t_2, t_3, t_4, t_5 may be drawn. These together with the first form one of the ∞^1 pentagons inscribed to the Lüroth curve. By means of such a pentagon both curves may be represented in the well known manner:

$$C_4 = \Sigma a_i t_i^4 = 0, \qquad S = \Sigma b_{ijkl} t_i t_j t_k t_l = 0.$$

THE UNIVERSITY OF ILLINOIS

ON A CERTAIN METRIC ASPECT OF PLANE PROJECTIVE TRANSFORMATIONS*

BY LULU HOFMANN

1. *Introduction and Results.* This paper contains the study of a special type of metric relations between projective planes. To bring out clearly the point of view from which the planes are considered, we first briefly summarize the principal known facts about the subject in general, remarking that by "metric" relations we refer merely to properties based on the ordinary euclidean measure of a straight line segment and an angle.

Let π and π' be two projectively related planes.† At a general point of $\pi(\pi')$ there are ∞^1 angles and on a general line ∞^1 segments which preserve their size under the transformation. These angles and these segments form an involution at the point and on the line.

There exist, however, in π and π' two distinguished pairs of corresponding points, the focal points, and two distinguished pairs of corresponding lines, the equi-segmental axes, such that all corresponding angles at these points and all corresponding segments on these lines are congruent. In either plane, these points and lines lie symmetrically with regard to the vanishing line. If the distance of the focal points from the vanishing line is c in π and c' in π', then the distance of the equi-segmental axes from this line is c' in π and c in π'. The quantities c and c' are called the parameters of the transformation, and the line joining the focal points in either plane is called the focal axis.

Thus in two projective planes, the metric properties of

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† For the following see H. J. S. Smith, *Memoir on the focal properties of homographic figures*, Collected Works, vol. 1, p. 545–602. We assume with Smith throughout this paper that the projective transformation considered is a general one, and we exclude the affine case.

corresponding segments on corresponding lines and of corresponding angles at corresponding points are known. In this paper, we start out from a new combination of elements, namely we study the metric properties of *corresponding segments at corresponding points.* We choose an arbitrary pair of corresponding points P_0 and P_0' in π and π' and examine those segments radiating out from P_0 which preserve their length as they are transformed into segments radiating out from P_0'. In other words, *after choosing a fixed pair of corresponding points P_0 and P_0', we determine the locus of the points P of π such that the segment PP_0 is congruent to the transformed segment $P'P_0'$.*

The locus of the points P of π such that

$$PP_0 = P'P_0' \tag{1}$$

where P_0 is fixed, is found to be a bipartite quartic Q of genus one. Its two branches* lie on opposite sides of the vanishing line; they intersect only at the infinite point of this line, thus making it a double point, and each there has the corresponding equi-segmental axis for its asymptote.

The second double point of Q lies at the fixed point P_0. It is a node, a cusp, or an isolated double point, according as there are two, one, or no real line elements at P_0 which are equal to their corresponding elements at P_0'. The points of π belonging to the second class form a bipartite sextic which separates those of the first from those of the third class.

The quartic Q is tangent to the circles about P_0 as center at the cyclic points. On any such circle there are therefore four, and on a general algebraic curve of the nth order† $4n$ finite points obeying condition (1).

* For the sense in which we use the word *branch* we refer, for example, to the definition of *complete branch* given by H. G. Zeuthen in the first paragraph of his paper *Sur les différentes formes des courbes planes du quatrième ordre,* Mathematische Annalen, vol. 7 (1874), pp. 420–432. In translation: "We denote by *complete branch* the part of a plane curve which can be described in a continuous manner by a point which, after having passed once through every fixed point, returns to its initial position."

† A curve not passing through the cyclic points and the infinite point of the vanishing line.

Aside from all metric considerations, Q may be described in a purely projective manner as the locus of the ∞^1 double points determined by the projectivity of π and π', as these planes are laid onto one another in the ∞^1 ways possible such that P_0 and P_0' coincide.

The extension to the case where corresponding segments at corresponding fixed points are required not to be equal but to form an arbitrary fixed ratio $k>0$, so that $PP_0 = k \cdot P'P_0'$, furnishes nothing new. It is identical with the original problem for the transformation with the parameters c and $\sqrt{k} \cdot c'$.

2. *Fundamental Properties of the Locus Q, its Double and Infinite Points, its Real Asymptotes, and its Branches.* We represent the projective relation between π and π' in the canonical form,* choosing the focal axis and the vanishing line in either plane as the xy and $x'y'$ axes. The transformation formulas then are

$$x' = \frac{cc'}{x}, \qquad y' = \frac{c'y}{x}; \tag{2}$$

c and c' are the parameters of the transformation; the focal points of $\pi[\pi']$ have the coordinates $(\pm c, 0)$ $[(\pm c', 0)]$ and the equi-segmental axes the equations $x = \pm c'[x' = \pm c]$.

Let $P_0(x_0, y_0)$ and $P_0'(x_0', y_0')$ be the fixed pair of corresponding points. Then the locus of the points P such that

$$PP_0 = P'P_0'$$

is obtained by replacing in

$$(x - x_0)^2 + (y - y_0)^2 = (x' - x_0')^2 + (y' - y_0')^2$$

the primed coordinates by the unprimed ones according to the formulas (2) of transformation. After ridding the resulting expression of its denominators and writing it in the homogeneous coordinates x, y, t, instead of x, y, we obtain

$$x_0^2 x^2[(x - x_0t)^2 + (y - y_0t)^2] - c'^2t^2[c^2(x - x_0t)^2 + (xy_0 - yx_0)^2] = 0. \tag{3}$$

* See the above cited paper by Smith, p. 561.

The locus (3) *of the points P fulfilling condition* (2) *is a quartic curve.* We call it Q. It is obvious from (3) that Q *has double points at* (x_0, y_0) *and* $(x=t=0)$, *that is, at the fixed point* P_0, *and at the infinite point of the vanishing line* (*y-axis*). .

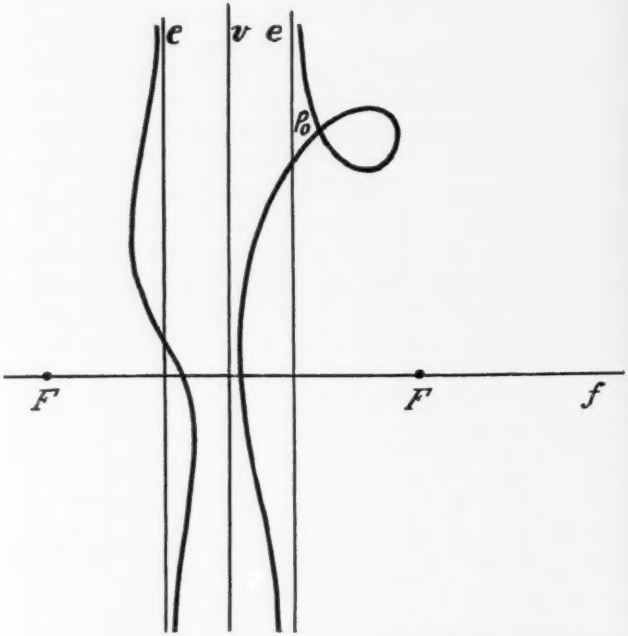

FIG. 1. SCHEMATIC FIGURE OF Q (Focal points F; focal axis f; vanishing line v; and equi-segmental axes e).

The substitution of $x = \pm c't$ into (3) furnishes two expressions with the factor t^3. It *follows that Q is tangent to the two equi-segmental axes at infinity and therefore always has a node at the infinite point of the vanishing line.*

Besides this node, Q has in common with the vanishing line the points $(0, \pm ic)$* and with the line at infinity the

* These points correspond to the cyclic points of π'.

cyclic points. Thus, since it does not have any real finite points on the vanishing line, nor any real infinite points except $(x=t=0)$, and since it has two real distinct asymptotes, parallel to and lying on opposite sides of the vanishing line, it follows that *Q consists of two branches separated by the vanishing line and intersecting only at the infinite point of this line.* Obviously *the branches are odd* (they have only one point of intersection).

If we consider a segment PP_0 not only with regard to its end points but with regard to all the points which make it up, we have to distinguish between the two branches of Q. For a segment with its end point P on that branch of Q which lies on the same side of the vanishing line as P_0, is transformed *point by point* to the segment $P'P_0'$; while, if P lies on the other branch, the segment PP_0 contains a point of the vanishing line and corresponds therefore point by point not to $P'P_0'$ itself but to the complementary infinite segment.

3. *The Intersections of Q with the Circles about P_0 and the Character of this Point as a Double Point. The Sextic S. Properties of Symmetry of Plane Projectivities.* It is obvious from the first term of (3) that *Q not only contains the cyclic points but is there tangent to the circles about P_0 as center.* It follows that Q has in common with any such circle four finite points; when the radius of the circle is sufficiently large, they are all real, as is evident from the character of Q. There is only one imaginary circle which intersects Q three times at each of the cyclic points.

If r is an arbitrary length, there are always exactly four points at this distance from P_0 which are transformed into points at the same distance from P_0'.

To find the four points of π belonging to a certain distance r, independently of Q, we invert for a moment the hitherto considered order of transformation, regarding π' as the original and π as the transformed plane. A circle of radius r about P_0' is transformed into a certain conic of π. The four points in question are found by intersecting this conic with the circle of radius r about P_0 itself.

If we take for r a vanishing quantity dr, the conic in π becomes a vanishing ellipse with P_0 as a center* and therefore has in common with the concentric circle of radius dr two real distinct, coincident, or imaginary diameters. This means *there are two real distinct, coincident, or imaginary line elements at P_0 which are equal to the corresponding elements at P_0', and accordingly P_0 is a node, a cusp, or an isolated double point for Q.* We denote the points of the three different types as N-, C-, and I-points.

We determine the character of P_0 analytically, by demanding that it be a triple point of intersection with Q for the line $y-y_0=\alpha(x-x_0)$ and investigating when, under this condition, α assumes two real distinct, coincident, or imaginary values. Calculation shows that these cases occur according as

$$(4) \qquad c'^2x_0^4t_0^2 + c'^2x_0^2t_0^2(y_0^2 + c^2t_0^2) - x_0^6 - c^2c'^4t_0^6$$

is greater than, equal to, or less than zero.

Expression (4) equated to zero and considered as an equation in x_0, y_0, t_0† shall be referred to as the sextic S. S is the locus of the C-points and separates the N-points from the I-points.

For the intersections of S with the focal axis we have $x_0=\pm c'$, $\pm(cc')^{1/2}$, $\pm i(cc')^{1/2}$. Let us denote by Γ and γ the greater and smaller of the quantities c and c'. It is easily seen that both when $|x_0|<\gamma$ and when $|x_0|>\Gamma$,(4) is negative, while it is positive when $|x_0|=c'$, $y_0\neq 0$. This indicates the division of π into the regions N and I as shown in Fig. 2.

We easily deduce the main properties of S by equating (4) to zero. *S is symmetric with regard to the vanishing line and the focal axis.* At the infinite point of the vanishing line, it has four coinciding points of intersection with a general line, and six with the vanishing line itself and with

* Disregarding infinitesimals of higher order than dr. See Smith, p. 559.

† We have again replaced the non-homogeneous coordinates by homogeneous ones. The suffix zero merely indicates that the respective point appears in the character of the fixed point of our problem.

the focal axis. Thus these two lines are the only tangents of S at its quadruple infinite point. Referring to the symmetric character of the curve, one can show that this point, besides being the only infinite one is also the only multiple one of S. It follows that *S consists of two branches which each have a point of self contact at the infinite point of the vanishing line. The first branch intersects the focal axis at the points $x = \pm\gamma$ and is tangent at infinity to the vanishing line. The second intersects the focal axis at the points $x = \pm\Gamma$ and is tangent at infinity to the line at infinity itself.*

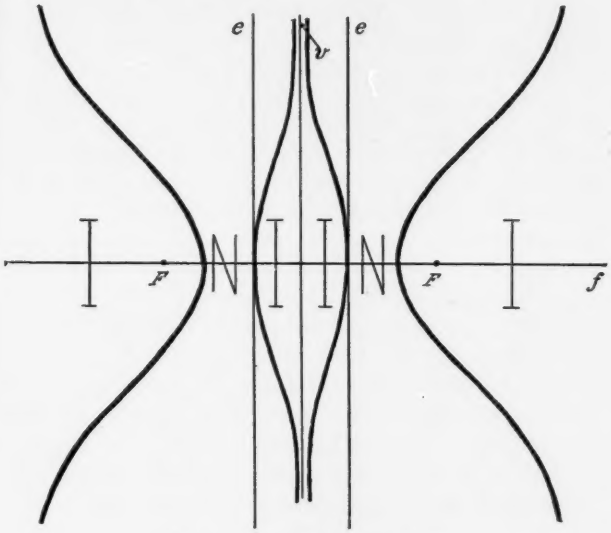

FIG. 2. SCHEMATIC FIGURE of S.

We have stressed the symmetric character of S. We add the remark that *two curves $Q^{(1)}$ and $Q^{(2)}$ belonging to points $P_0^{(1)}$ and $P_0^{(2)}$ which lie symmetrically with regard to the vanishing line or the focal axis, are mutually symmetric with regard to the respective line.* This includes as a special statement that *the Q-curve belonging to a point P_0 on the focal*

axis, is itself symmetric with regard to this axis. The metric point of view brings out the symmetric properties of plane projectivities.

4. *Interpretation of Q as π and π' Coincide and Proof that it is of Genus One.* Let π' be laid on π in such a manner that P_0 and P_0' coincide. There will be two more points, say P_1 and P_2 of π which coincide with the corresponding points of π', thus with P_0 forming the three double points of the transformation. By definition, P_1 and P_2 lie on Q; for, being a double point of the transformation, $P_1=P_1'$ ($P_2=P_2'$) as a point of π has the same distance from P_0 that it has from P_0' as a point of π'. Thus, *Q is the locus of the ∞^1 double points determined by the projectivity as π' is laid on π in the ∞^1 ways possible such that P_0 and P_0' coincide.*

Choosing one definite such position of π and π' with the double points P_0, P_1 and P_2, let us consider the line l through P_0 and P_1. It cuts Q in a fourth point (P_0 counting as two), say P_1^*. The point P_1^* belongs, as a double point of the transformation, to that position of π and π' which is obtained from the one considered by rotating π' through an angle of 180° about the perpendicular to l which lies in π and passes through P_0. Accordingly, P_1 and P_1^* can coincide only at the point P_0 or at infinity. On the other hand, the coincidence of P_1 and P_1^* is equivalent with the existence of a double point of Q, so that double points of Q can occur only at P_0 and at infinity. But we know that Q has only one infinite double point, namely, at $x=t=0$. Thus we have proved that *Q has exactly two double points and is therefore of genus one.*

5. *Degenerations of Q and Specializations of S.* To study the cases in which Q degenerates, we let P_0 assume all possible positions excepting those on the vanishing line and the line at infinity. The character of our problem makes it natural to consider only such finite points P_0 which are again transformed into finite points P_0'.

Owing to the fact that Q consists of two branches lying on opposite sides of the vanishing line and each having one of

the equi-segmental axes for its asymptote, the only possible degenerations of the curve are those in which it contains one or both of these axes as a part. Calculation shows that in the first case P_0 lies on one of the equi-segmental axes ($x_0 = {}^{+}_{(-)}c'$) and in the second case at one of the focal points ($x_0 = {}^{+}_{(-)}c$, $y_0 = 0$). The corresponding equations for Q are

$$(x_{(}\mp_{)}c't)\left[(x_{(}\mp_{)}c't)(x^2 - c^2t^2) + y\left\{x(y - 2y_0t)_{(}\pm_{)}c'yt\right\}\right] = 0,$$
$$(x^2 - c'^2t^2)\left[(x_{(}\mp_{)}ct)^2 + y^2\right] = 0.$$

When P_0 lies on one of the equi-segmental axes, Q consists of the respective axis and a cubic, which is made up of an oval through P_0 and an odd branch with the other axis for its asymptote. The branches lie on opposite sides of the vanishing line and contain the respective focal points.

When P_0 lies at one of the focal points, Q consists of the absolute lines through this point and both equi-segmental axes.

The degenerations of Q are caused by special positions of P_0 and occur in every projectivity, that is for every value of the parameters c and c'. S, however, depends on these parameters only. It is evident from its constitution that it cannot degenerate, but that it specializes when $c = c'$. In this case it becomes unipartite and adds to its quadruple singularity at infinity two nodes at the focal points. On the contrary, the equality of c and c' has no essential effect on a general Q-curve.

6. *Extension of the Problem to the Case of Corresponding Segments forming a General Fixed Ratio $k > 0$.* We have so far investigated the locus of those points of π whose distances from a fixed point P_0 remain unchanged under the transformation. The investigation immediately extends itself to the case in which we require these distances to be changed according to any fixed ratio $k (k > 0)$: $PP_0 = k \cdot P'P_0'$. Then (3) assumes the form

$$(3)' \quad x_0^2 x^2\left[(x - x_0t)^2 + (y - y_0t)^2\right]$$
$$- kc'^2t^2\left[c^2(x - x_0t)^2 + (xy_0 - yx_0)^2\right] = 0.$$

But this is the equation corresponding to the original problem ($k=1$) for a projectivity with the parameters c and $\sqrt{k}\cdot c'$. The new curves Q_k and S_k are therefore of exactly the same type as Q and S. The ∞^1 curves Q_k belonging to the same point P_0 form a pencil whose base points are P_0 itself and the infinite point of the vanishing line, each counted four times, and the cyclic points and the points $(0, \pm ic)$, each counted twice.

7. *Summary.* In conclusion, we repeat the fundamental results of the preceding discussion in a form no longer distinguishing π before π', but stressing the obvious symmetry of the problem with regard to both planes.

In two projective planes there is associated with every fixed pair of corresponding points and every value of a positive quantity k a pair of bipartite quartics of genus one corresponding in the projectivity which are the loci of all the pairs of corresponding points whose distances from the respective fixed points form the ratio k. Moreover there is associated with every value of k a pair of bipartite sextics corresponding in the projectivity which are the loci of all those pairs of corresponding points at which there exists one real pair of corresponding line elements forming the ratio k. These sextics separate the points of their planes where there are two such pairs from the points where there is none.

The problem discussed in this paper immediately suggests two new ones. First, the problem corresponding by duality: the study of the pairs of corresponding lines forming congruent angles with two corresponding fixed lines. Secondly, the extension to spaces of higher dimensions. These problems lead to curves of the fourth class and hypersurfaces of the fourth order, but do not appear to furnish essentially new results.

COLUMBIA UNIVERSITY

LANDAU ON NUMBER THEORY

Vorlesungen über Zahlentheorie. By Edmund Landau. Volume I: *Aus der elementaren und additiven Zahlentheorie.* xii+360 pp. Volume II: *Aus der analytischen und geometrischen Zahlentheorie.* vii+308 pp. Volume III: *Aus der algebraischen Zahlentheorie und über die Fermatsche Vermutung.* vii+341 pp. Leipzig, S. Hirzel, 1927.

These three excellently printed and arranged volumes form an addition of the highest importance to the literature of the theory of numbers. With them, the reader familiar with the basic elements of the theory of functions of a real and complex variable, can follow many of the astonishing recent advances in this fascinating field. His interest is enlisted at once and sustained by the accuracy, skill, and enthusiasm with which Landau marshals the analytic facts and simplifies as far as possible the inevitable mass of details.

Part I of Volume I gives a treatment of the elements of the theory of numbers up to the theory of quadratic residues and of the Pellian equation. This occupies a little over 50 pages. In Part II is first deduced the elementary inequality

$$a_1\xi/\log \xi < \pi(\xi) < a_2\xi/\log \xi$$

for the number of primes $\pi(\xi)$ not greater than ξ, after which follows the proof of the theorem of Brun (1919) that, if the number of "twin primes" p, $p+2$ is infinite, these are at least so infrequently distributed that the series of reciprocals

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\cdots$$

converges, in contradistinction with the series of the reciprocals of all primes, which diverges. The reader is then introduced in the same part II to characters $\chi(a)$ mod k, and the associated Dirichlet series, by means of which is established Dirichlet's classical theorem that every arithmetical progression $l+ks(s=1, 2, \cdots)$ contains infinitely many primes if l and k are relatively prime.

All of this material in Parts I and II, which acquaints the reader with some of the most interesting ideas in number theory, occupies slightly less than 100 pages.

In Part III are obtained the classical theorems concerning the representation of an integer as the sum of two, three, or four squares, and in Part IV are developed the classical results concerning the class number of binary quadratic forms.

Part V presents that part of the recent extraordinary series of researches, *Some Problems of "Partitio Numerorum,"* Parts I–V (1920–1925), by Hardy and Littlewood, which lies in the direction of Goldbach's conjecture that every even integer >2 can be expressed as the sum of two odd

primes. *Under the hypothesis* that not only Riemann's function $\zeta(s)$, but all of the similar functions defined by Dirichlet series, $L(s)$, have no zero for s with real part $>\Theta$ with Θ a definite constant $<\frac{3}{4}$, it is proved that every sufficiently great odd integer is representable as the sum of three odd primes, and that "almost all" even positive integers are the sum of two odd primes.

The sixth and last part of Volume I develops the part of the results of Hardy and Littlewood bearing on Waring's theorem (first proved by Hilbert, 1909), that any integer can be expressed as the sum of a fixed number g of nth powers, $g=g(n)$, in which they go a considerable distance toward determining the nature of the number theoretic function $G(n)\leqq g(n)$ representing the number G of powers sufficing for all large integers, and the function $G_1(n)\leqq G(n)$ representing the number G_1 sufficing for almost all large integers; for example, it is proved that

$$G(n) \leqq (n-2)2^{n-1}+5.$$

It is interesting that Landau's beautiful earlier result $G(3)\leqq 8$ (1909) escapes the analysis of Hardy and Littlewood who prove only $G(3)\leqq 9$.

At the end of this part is found Vinogradoff's brief proof (1922) of Waring's theorem, involving the use of Weyl's approximation theorems (1916), occupying in all less than 20 pages.

In Part VII, Volume 2, the principal result (Littlewood, 1924) is that the difference between the prime number function $\pi(x)$ and $\int_2^x (du/\log u)$ is of the order of

$$xe^{-\alpha(\log x\log\log x)^{1/2}}$$

at most. This result, as well as the analogous fact for any $\pi(x; k, 1)$ is proved in a direct and simplified manner by Landau. Here the comparison with the earlier result of de la Vallée Poussin that this difference is of the order

$$xe^{-\alpha(\log x)^{1/2}}$$

at most, measures the progress made since the appearance of Landau's *Handbuch der Lehre von der Verteilung der Primzahlen* (1909).

Part VIII of the same volume takes up thoroughly the number theoretic function $A(x)$, defined as the number of integer pairs (u, v) for which

$$u^2+v^2 \leqq x,$$

that is, the number of points with integer cartesian coordinates within or on the boundary of the circle of radius $x^{1/2}$ and with center at the origin in the u, v plane. The fact that $A(x)-\pi x$ is of the order of $x^{1/2}$ at most is trivial. The major question is to determine as far as possible the order θ of this difference, the above trivial result showing that $\theta\leqq 1/2$. Here the facts established are $\theta\geqq 1/4$ (Hardy and Landau, 1915) and $\theta<1/3$ (Sierpinski, 1906; van der Corput, 1923). Similar results are deduced for other curves.

The first part of Volume 3 (Part XI) expounds the classical theory of algebraic numbers and ideals. Here the principal new development

to be found is the proof of Thue's theorem (1908) that if $g(x, y)$ is an irreducible homogeneous polynomial of degree $n>2$ with integral coefficients, the equation $g(x,y)=a$ for a given integral value of a has only a finite number of solutions. The main purpose of this part is to develop as much of the theory of ideals which is necessary to establish the known results concerning Fermat's famous conjecture that the equation

$$x^n + y^n = z^n$$

has no integral solution x, y, z with $xyz \neq 0$ for $n>2$. In particular the proof of Kummer's theorem that the conjecture holds for $n=p$, a regular prime, is given in full. The concluding part deals with some recent results of Furtwängler (1912), Wieferich (1909), Mirimanoff (1910), and Vandiver (1914, 1919).

The mathematical world owes a great debt of gratitude to Professor Landau for rendering accessible so many of the recent splendid achievements in the theory of numbers.

G. D. BIRKHOFF

THREE BOOKS ON WAVE MECHANICS

Four Lectures on Wave Mechanics. By Erwin Shrödinger. London and Glasgow, Blackie and Son, 1928. viii+53 pp. Price 5 s.

Collected Papers on Wave Mechanics. By Erwin Schrödinger. Translated from the 2d German edition by J. F. Shearer and W. M. Deans. London and Glasgow, Blackie and Son, 1928. xiii+146 pp. Price 25 s.

Selected Papers on Wave Mechanics. By Louis de Broglie and Léon Brillouin. Translated by W. M. Deans. London and Glasgow, Blackie and Son, 1928. 151 pp. Price 15 s.

The wave mechanics proposed by de Broglie and Schrödinger has assumed during the short space of two years such a dominant role in the field of atomic physics that these three books in English by the authors of the new theory will be eagerly welcomed by physicists and mathematicians in England and America.

Schrödinger's *Four Lectures on Wave Mechanics* were delivered at the Royal Institution last March. In them are explained the ideas underlying the wave mechanics and some of the more important applications of these ideas. The exposition is somewhat popular in form, the mathematical details of computation being omitted from the text. The first lecture unfolds the relation between geometrical optics and classical dynamics as exhibited by the correspondence between Fermat's principle and the principle of Maupertuis (least action). It is this correspondence which suggests that classical methods are no more applicable to micro-mechanics than the ray methods of geometrical optics are to problems in diffraction and which leads in atomic problems to the substitution of a wave equation for the Hamilton-Jacobi equation of macro-mechanics. The success of a physical theory is measured by the agreement of its predictions with experi-

ment, and in the succeeding lectures applications of the theory are made to the hydrogen atom (including the case of a nucleus of finite mass), the simple harmonic oscillator and the rotator. The physical interpretation of the ψ function proposed by the author is shown to lead to correct selection and polarization rules and to correct intensity ratios for the emitted radiation. Finally, the perturbation theory is developed and from it the dispersion formula is deduced.

The serious student of the new mechanics will find a complete account of the subject in the *Collected Papers*. These comprise practically all the author's papers on wave mechanics, including the one in which he shows the equivalence of his theory with the matrix dynamics of Heisenberg, Born and Jordan. With one exception all of them appeared in the *Annalen der Physik* and all were published during the two years 1926 and 1927. As the author remarks in his preface, the results contained in the later papers were unknown to him when the earlier ones were written, and consequently the development is not as logical as it might be made if the book were rewritten in the light of his present knowledge. Nevertheless his exposition of the subject will probably appear clearer to the reader than those contained in several books which have been hurriedly thrown together by others in the attempt to supply the demand for information on the new theory.

Schrödinger's ideas originated in those of de Broglie, who was the first to suggest that material particles might have the properties of a wave group. The *Selected Papers on Wave Mechanics* include several of the fundamental papers of de Broglie as well as papers by Brillouin on both the matrix dynamics of Heisenberg and Dirac and the wave dynamics of de Broglie and Schrödinger. The last paper in the series contains a valuable comparative study by Brillouin of the classical Bose-Einstein and Fermi-Dirac statistics.

These three volumes taken together provide an excellent exposition of those theories of atomic physics which have proved most successful up to the present time. The translations of the last two have been carefully prepared, the originals having been followed as closely as English idiom would permit.

LEIGH PAGE

FRAENKEL ON GRUNDLEGUNG DER MENGENLEHRE

Zehn Vorlesungen über die Grundlegung der Mengenlehre. (Wissenschaft und Hypothese, XXXI). By Adolph Fraenkel. Leipzig and Berlin, B. G. Teubner, 1927. x+182 pp.

This book embodies a series of ten lectures delivered in Kiel, June 8–12, 1925, at the invitation of the *Kant-Gesellschaft, Ortsgruppe Kiel.* The objects to be accomplished by its publication seem to be two-fold: (1) To render intelligible to non-workers in the field, especially philosophers, past and present views on the "Foundations," and (2) to stimulate reflection amongst the mathematical public concerning the foundations of mathematical thought and methods.

With regard to (1) it seems to the reviewer that the author has been singularly successful. We have here in no sense a formal text, the informal nature of the lectures having been preserved, with desirable additions made as a result of the subsequent discussion, together with literary references, etc. And yet the treatment is not superficial. The author plunges at once into a discussion of the notions of "set", equivalence of sets and cardinal numbers, order types, the continuum problem, the "diagonal procedure", antinomies and paradoxes. Of the last are given the (Zermelo-) Russell Paradox, the Burali-Forti Antinomy and Richards' Paradox. The consequences of the antinomies in mathematical thought are briefly mentioned—the failure to give a definition of the "set" notion less "offensive" than the Cantorean, in whose looseness seems to lie the germ of such paradoxes as Russell's—the suggestion that the fault be ascribed to logic, not to mathematics—the resignation of such men as Dedekind and Frege to what they considered the inevitable results of the antinomies.

The remainder of the lectures is devoted to the efforts of various schools so to revise the guiding rules of mathematical technique that the antinomies cannot arise. An exposition is given of the views of the Brouwer school (Intuitionism), as well as of the modified intuitionism of Poincaré, and the changes in the structure of the mathematical edifice that follow an adoption of the intuitionist views. There are also references to the system of Russell and Whitehead.

By far the greater part of the book (from page 58 on), however, is given over to the axiomatic foundation of the Mengenlehre begun by Zermelo. A pleasing feature of this part of the work is the way in which the author leads up to outstanding unsolved problems—certainly the mathematician who reads will *feel* the urge to apply himself to their solution, whether or not he actually yields to the temptation. Another pleasing feature is the occasional discussion of much debated matters, such as the "choice axiom", the non-predicative procedure and the excluded third. The axioms are seven in number. One of these is the choice axiom. Another axiom definitely implies the use of the logical law of the excluded third. And the non-predicative method seems to be indispensable for the axiomatic development.

The author's justification of these inclusions is exceedingly interesting from any point of view, be it philosophical, mathematical or purely logical. Thus, in regard to the choice axiom, after a historical summary of its use, its formulation, and criticisms directed against it, the author maintains "This logical principle has certainly at least a character of evidence and necessity equivalent to that which one attributes to many other axioms that one is accustomed to recognize as indispensable for the foundations of arithmetic, analysis and geometry; it was approved even by one so close to the intuitionist viewpoint as Poincaré. With equal right, then, to that with which one rejects the choice axiom, one could arbitrarily deny other important fundamental principles, and so in the future ban important parts of mathematics." The placing of the choice principle in the axioms is then compared to the assumption of the parallel axioms by the Greeks. This analogy leads the reviewer to wonder if the denial of the choice principle might not lead to fruitful results as did the denial of the parallel axiom. A non-euclidean geometry proved to be a happy conception, and might there not possibly be some significance in a non-Zermeloan Mengenlehre which asserts the existence of sets that cannot be well-ordered?* On a historical basis, at least, one would seem justified in maintaining that the denial of "important fundamental principles" is not necessarily an iconoclasm.

What would probably comprise the tenth lecture (the division of topics occurs in pairs of lectures) is devoted to a discussion of consistence, independence and categoricalness of axiomatic systems in general as well as of the particular system presented for the Mengenlehre. Although the system which Fraenkel gives does away with the existing paradoxes and antinomies, its consistency is an open question. From the point of view of him who accepts the axiomatic foundation this is a great defect—indeed, it can be said that if ever a consistency proof were needed, it is needed in this particular instance, since the very purpose of the adoption of the axiomatic method in this case is to avoid contradiction.

The reviewer heartily recommends these lectures to any who wish a general summary of the state of the "Foundations," as well as to those who wish a good introduction for mature students. The book is finished off with an unusually complete bibliography comprising eight pages, and this is amplified by up-to-date additions made during the period of proof-reading.

R. L. WILDER

* See Church, *Alternatives to Zermelo's assumption*, Transactions of this Society, vol. 29 (1927), p. 178.

GILLAIN ON EGYPTIAN SCIENCE

La Science Égyptienne. L'Arithmétique au Moyen Empire. By O. Gillain, avec une préface de H. Bosmans, S. J. Brussels, 1927. xvi+326 pp.

This work is one of the publications of the Fondation Égyptologique Reine Élizabeth, and like the other works bearing the same imprint it represents a standard of scholarship as well of typography which commends it both to students and to general readers. Not the least of its merits is the preface written by the late and widely lamented Père Bosmans, a scholar and a "gentleman of the old school," whose knowledge of the history of mathematics, as also of medieval ecclesiastical literature, was the result of the labors of a long and useful life.

The title of the book does not indicate very clearly the nature of the work, which is chiefly a treatise upon and quite largely a translation of the Rhind (Ahmes) Papyrus, but with a brief consideration of other manuscripts in the Introduction and in Chapter IV. In one way such a work may be thought to be superfluous in-as-much as we have two editions of Eisenlohr's translation, the much better edition of Peet, and the still better and more elaborate one by Dr. Chace, the second volume of which is just appearing. In another sense, however, it is to be welcomed, since it gives to French readers who may not be familiar with German or English a good idea of the oldest mathematical treatise of any extent that has come down to us.

Since the nature of this papyrus is now so well known to American and British readers, both through the editions above mentioned and through recent reviews, it is sufficient at this time to call attention to the general line of treatment followed by M. Gillain. His Introduction (pp. 1–22) sets forth the nature of Egyptian mathematics of the period, describes the measures then in use, mentions the fantastic theories of the Pyramid measurers, and concludes with a brief history and description of the work of Ahmes. He also refers to the earlier fragments of a mathematical nature, including the Moscow manuscript made known by Turaev (Touraieff) in 1917, and soon to be published in translation. As to the Pyramid theorists, M. Gillain expresses the very sane opinion, "La Grande Pyramide est sans contredit, admirable, mais ses constructeurs n'étaient point sorciers."

The work is divided into four chapters, as follows: Chapter I,—Elementary computations, in which are explained the general methods used in what we call the four fundamental operations, both with integers and with fractions, illustrative examples being selected from the Ahmes text. Chapter II,—The tables of fractions, in which a comparison is made between the cumbersome method used by the Egyptians in, say, dividing 37 by $1+2/3+1/2+1/7$,—a problem which the Egyptian calculator might have had to solve,—and the simple division of 37 by the equivalent fraction 97/42, which might conceivably be required in our day. This is followed by the usual tables of $2/(2n+1)$ for values of n from 1 to 50, and by examples from the text illustrating its use. Chapter III,—Problems, in which the

author considers such topics as the rule of three, false position, proportion, progressions, and roots. Chapter IV,—The spirit of arithmetic, in which he briefly discusses other sources than the Rhind Papyrus and considers Egyptian calculation in general and the nature of the Egyptian mind as revealed by the arithmetical sources known to us. His conclusion is clear and succinct,—"Telle que nous la connaissons, l'arithmétique égyptienne n'est ni plus ni moins qu'une physique des nombres,"—it never reached the metaphysical stage.

The chief merit of the work lies in the fact that the author has set forth, for French scholars, the general nature of Egyptian arithmetic in about the 17th century B.C., or a little before, illustrating his points by translations. These translations are chiefly if not wholly from the Peet edition of the Rhind Papyrus. The interpretation of Ahmes and of Egyptian arithmetic in general, shows a considerable range of reading from such standard writers as Bobynin, Eisenlohr, Favaro, Griffith, Gunn, Hultsch, Loria, Neugebauer, Révillout, Schack-Schackenburg, and Sethe.

As to the interpretation of the obscure passages in Ahmes, the student will do better to consult Dr. Chace's monumental work, and as to a bibliography of the subject M. Gillain's list is not in the same class as the remarkable one by Professor Archibald which is published in volume I of the same treatise.

One thing that frequently strikes the reader of scientific works in French, rather more than in other languages, is the carelessness shown in proofreading. Possibly the trouble is primarily due to French chirography, which renders it difficult for compositors to distinguish between certain letters; possibly the French scholar sacrifices niceness of typographical detail to niceness of expression; and possibly the habit of author proofreading, which did not characterize the early products of the press has not been as fully established in the Gallic regions as in those under Teutonic or British intellectual influence. Whatever may be the cause, the work of M. Gillain suffers, like so many other works in the same language, from this defect. Thus we have such spelling as Karpinsky for Karpinski, Piazzi Smith for Piazzi Smyth, Schackenbourg as well as Schackenburg, Brunschwigg for Brunschvicg, together with Sepher ha-mmispar and Gottingsche, to select only a few examples. These are mentioned not primarily as criticisms of M. Gillain's work, but rather as a problem in race psychology which is here illustrated. As already stated, the work itself has distinct merit and will be very helpful to French and Belgian students.

DAVID EUGENE SMITH

SHORTER NOTICES

Modern Researches on the Singularities of Functions Defined by Taylor's Series. By S. Mandelbrojt. Edited by E. R. C. Miles. The Rice Institute Pamphlet, Volume XIV, No. 4, pp. 225–352.

The material collected in this pamphlet is a result of a series of lectures delivered by the author at the Rice Institute during the academic year 1926–27. The purpose and scope are best described by a short paragraph taken from a foot-note on the first page of the book:

"The present type of research, which began with the famous thesis of Hadamard, has broadened considerably during the past few years. Complete proofs of all the theorems cannot be compressed in the space available in a short treatise, so the author treats only of such theorems as seem to give unity to the theory, and of the latter theorems proofs are given, for the greater part, in detail. The reader will find a large bibliography, as well as an enumeration of nearly all the results on the singularities of Taylor's series in Hadamard and Mandelbrojt: *La Série de Taylor et son Prolongement Analytique*, Scientia, No. 41, also in the author's volume of the Mémorial des Sciences Mathématiques. The present treatise may be regarded as complementary to, as well as an elaboration of, some parts of the works just mentioned."

The first four chapters deal with that part of the theory which has now become classical, treating such material as the theorems of Hadamard and Hurwitz on the composition of singularities and a necessary and sufficient condition that a function should be meromorphic on its circle of convergence. In Chapters V and VI the author launches into the more recent developments of the theory, proving first an interesting theorem of his own regarding power series which have on their circle of convergence at least one singularity not a pole, and then giving various generalizations and applications. Chapter VI again returns to classical theory, presenting the theory of order of singularities. In criticism of this chapter we call attention to the fact that although the first section defines and discusses fractional differentiation and integration, no mention is made of its connection with the Hadamard operator defined in the second section. A reader meeting the subject for the first time (and it is believed that the book will in general be valuable for such readers) may naturally inquire why fractional differentiation is introduced at all. It is not until the close of the chapter, twenty pages farther along that one reads the statement, without proof, that if one of these operators yields a continuous function of finite deviation so too does the other. Chapters VIII and IX deal with the fundamental theorems of Faber and Fatou and with the author's generalizations of the latter. Chapters X and XI center around Dr. Mandelbrojt's contributions to the theory and related theorems. The final chapter deals with series having the circle of convergence as a cut, following the method introduced by Hadamard and amplified by Fabry, Leau and others.

The reviewer believes that the book will be a welcome addition to the treatises on the subject. Mr. Miles and those responsbile for the Pamphlet are to be congratulated upon the careful way in which the lectures are edited. Misprints are rare, and the general high character of the Pamphlet, established in earlier numbers dealing with mathematical subjects, is maintained. The consistency with which references are given to unproved theorems and to the proofs of facts employed in the demonstrations is highly commendable. If this principle were followed more generally, mathematical literature would be more readable.

D. V. WIDDER

Theory of Probability. By William Burnside. Cambridge University Press, 1928. 106+xxx pp. Short table of Error-Integral.

This *Theory of Probability* was edited for the press by A. R. Forsyth from a manuscript which Burnside practically completed before his death. In barely a hundred pages, it treats a great variety of topics in a decidedly unique and interesting manner. It is not a book on statistics. Burnside's interest is in probability itself. First, he develops in somewhat extended form relations between probabilities of a general nature associated with n conditions, some of which are to be satisfied and some not—connecting his primal idea of equal likelihood with these conditions. Then he proceeds to the discussion of typical problems, mainly algebraic in their origin. Analysis is freely used—of course, it is indispensible—but the reader gets the impression that Burnside tries to view a problem first in an algebraic setting. Thus the difference equation is more in evidence than the differential equation. This is a distinctive feature. It makes the book of special value to readers who have cultivated mainly the analytic side.

The book is fairly well organized, in spite of the fact that Burnside may not have considered it in its final form. A greater unification of topics involving the Gaussian law would have been desirable,—this law appears on pages 42, 44, 52, 73, 88. On page 73, the validity of this law as the best approximation is not so clear, as the elementary probability designated by x is made very small—this leading naturally to the Poisson Exponential, obtained on page 45. Burnside is not satisfied with the usual statement of the assumption of equal likelihood. He insists upon "assuming each two of the n results equally likely," instead of "assuming all the n results to be equally likely," see page 101. Burnside's phrasing appears a little more specific—although it seems difficult to conceive how all can be equally likely if some two of the results are not equally likely. In Chapter VII, which deals with the theory of errors, assumptions are given which lead respectively to the arithmetic mean, the median, and the average of the least and the greatest measurement. The arithmetic mean is found acceptable "as any other assumption would imply that either an excess of positive errors or an excess of negative errors has occurred." If here by "positive errors" the *number* of positive errors is meant, the median would be indicated rather than the arithmetic mean. The argument in Chapter VI follows conventional lines, including an artificial assumption regarding the a priori

probability at the top of page 85, made so as not to violate the first equation on page 84. Not many disturbing typographical errors were found—but the radicals at the foot of page 89 and at the top of page 90 should be in the denominators, and on page 91, the coefficient of the square of X should be inverted. On page 64 in the last line, $s/2$ should be replaced by $(s/2)-1$, and $(s/2)-1$ by $(s/2)-2$. Here and at the foot of page 98 mention might have been made of Pearson's *Tables of the Incomplete Gamma Function* for computational completion of the problems. At the foot of page 35 it is assumed that p does not equal 1/2.

To epitomize the content of Burnside's book is somewhat difficult. The last chapters deal with the probability of causes (with interesting exemplification of Bayes's Theorem), geometrical probability, and the theory of errors. As early as page 22, problems on "runs" appear, and these under the title "duration of play" constitute the principal part of Chapter III, and are again taken up on page 72. These problems are more fundamental than they at first seem. They are especially fascinating in a book so full of good material.

E. L. DODD

Drei Abhandlungen über die Auflösung der Gleichungen von Leonhard Euler (1738, 1764, 1790). Translated by Samson Breuer. 94 pp. Leipzig, Akademische Verlagsgesellschaft, 1928. Ostwald's Klassiker, Nr. 226.

Students interested in the history of the theory of equations will find here, reproduced in translation from the Latin into German, three treatises by one of the greatest mathematicians of the eighteenth century, who still held to the belief that the algebraic solution of the general quintic equation and equations of still higher degrees was possible. Knowing the extraordinary dexterity of Euler in analytical processes, we are impressed by his statement that though his article did not contain a complete solution, it indicated the form which a root must take, and perhaps it would be "of greater service to others and lead them finally to the desired goal." Euler continues: "Since an equation of any degree comprehends all equations of lower degree, the solution of an equation of any degree must involve all the processes for equations of lower degree." Euler assumed as given by induction that the resolvent of the general quintic is a quartic. In the second treatise Euler remarks that if p, q, r, s are the roots of the quartic resolvent, then the general form of a root of a quintic equation would seem to be $x=f+\sqrt[5]{p}+\sqrt[5]{q}+\sqrt[5]{r}+\sqrt[5]{s}$, but he remarks that this form is not sufficiently sharp as it would seem to lead to more than five roots of the quintic. Euler assumes next $x=f+A\sqrt[5]{p}+B\sqrt[5]{p^2}+C\sqrt[5]{p^3}+D\sqrt[5]{p^4}$, and is led to solvable equations, including those previously solved by De Moivre. In the third article, written in 1776 and printed in 1790, Euler assumed the coefficients to depend upon two or more parameters and then derived numberless forms of equations of all degrees, which are solvable by algebra. Breuer, the translator of these articles, has supplied much bibliographical detail and many notes aimed to assist the reader.

FLORIAN CAJORI

The Logic of Modern Physics. By P. W. Bridgman. New York, Macmillan, 1927. 14+228 pp.

The author begins by calling attention to the fact that the accepted classical concepts in physics were found inadequate to explain the experimental facts obtained in testing Einstein's deductions from his relativity theory. He points out the necessity of studying our concepts to prevent a similar failure in the future. By concept he means a set of operations. A lengthy study of the operations involved in measuring distances follows and the concept of an astronomical and an atomic distance is arrived at in terms not only of actual operations but also in terms of hypotheses involved. Following this mode of thought, concepts of space, time, identity, velocity, force, mass, energy, electricity, etc. are discussed in detail. In connection with light, the author concludes that "from the point of view of operations, it is meaningless or trivial to ascribe physical reality to light in intermediate space, and light as a thing travelling must be recognized to be a pure invention." It must follow that finite velocity of light does not exist and likewise Einstein's definition of simultaneity. This necessarily results from a conception that a reality can be only when a phenomenon exists. Difficulties arise which appear as great as those which phenomenalism tries to remove. Finally, the last chapter is devoted to a study of Special Views of Nature in which he is led to accept the postulate that physical laws are formally simple.

The book is an earnest attempt at formulation of concepts in physics based on operations with due regard for the hypotheses that may be involved. It should be found welcome to mathematicians and philosophers as well as experimental physicists.

A. F. Kovarik

Quellenhefte zur Elementarmathematik: Band I, *Elementargeometrie*, Heft 1: *Der Stoff bis zur Untersekunda.* By Kuno Fladt. Leipzig and Berlin, B. G. Teubner, 1928. vi+73 pp.

This is a source book on the history of geometry in which are reproduced selections from such parts of historic sources as have retained their place in elementary instruction down to the present time. No obsolete material is included. The booklet is a companion to another publication by the same author, written as a geometric text for pupils about fourteen years of age, in the Untersekunda. After a brief historical introduction there are given, in the source book, extracts, in German translation and in modern symbols, of such authors as Euclid, Archimedes, Proclus, Heron and some more modern writers, all accompanied with suitable historic notes. An attractive feature among the illustrations is the part of Raphael's School of Athens which shows the geometrical and astronomical group, and also the reproduction of a recently described mosaic, said to have been taken from the ruins of Herculaneum, and showing the death of Archimedes. The booklet represents an effort to utilize in elementary instruction the interest attached to the history of mathematics.

Florian Cajori

Einführung in die Theoretische Physick. By Max Planck. 1. *Einführung in die Allgemeine Mechanik.* 4th ed. viii+226 pp., 43 figures. R.M. 6—; bound, R.M. 8—. 2. *Einführung in die Theorie der Elektrizität und des Magnetismus.* 2d ed. viii+210 pp., 13 figures. R.M. 6—; bound, R.M. 8—. Leipzig, Hirzel, 1928.

These two volumes form the fourth and second editions respectively of two of the component parts of a work designed, when completed, to offer an introduction to the whole field of classical theoretical physics. The author's object is apparently to provide a series of texts which can be used with profit by the independent reader as well as by the university student. It would seem that in these two books the aim is well achieved.

The volume on mechanics is developed mainly from the standpoint of the author's pedagogical thesis that students beginning theoretical physics usually have less trouble with the mathematical discussion than with the physical interpretation. In line with this view the subject is not presented as a complete logical structure but the development is from the very simple to the more complicated. Elegant proofs of theorems give way to interpretative methods even at the expense of space. In formal construction the work is divided into the usual parts, viz. mechanics of a particle and mechanics of a system of particles. The treatment in the former case is not particularly noteworthy save that the author takes the concept of *force* as fundamental, ignoring the usual difficulties confronting the beginner in this connection. An interesting feature of the second part is the treatment of the Hamilton-Jacobi equation and its integration, so important in modern atomic theory.

The author has followed a somewhat different method in the volume on electricity and magnetism. There the treatment is severely logically deductive in nature rather than experimentally inductive. This is deliberately chosen to preserve unity of presentation and to emphasize the few fundamental concepts instead of the many diverse applications. The classical theory of Maxwell is followed with principal emphasis on the concepts of energy density and energy flow. Indeed the electric and magnetic fields are defined in terms of the former and the field equations are deduced by the application of the laws of Joule and Poynting and the conservation of energy. Electrostatics and magnetostatics are treated as special cases of the more general field relations (that is, no change with the time). Electric currents are treated as illustrations of "stationary" fields, viz. fields in which the electric and magnetic intensities do not change with the time but in which there is a continual change of electrical energy into heat. The "non-stationary" phenomena which form the general case of electrodynamics are treated in the last 60 pages of the book. Here the development follows classical lines very closely, and there is practically no mention of the electron theory. The difficulties of the original Maxwell-Hertz theory applied to moving media are merely mentioned and but a page or two devoted to Lorentz's solution.

The style is clear and the type is good. No serious errors have been detected. In both volumes the figures are rather poorly drawn.

R. B. LINDSAY

Einführung in die Axiomatik der Algebra. By H. Beck. Berlin, Walter de Gruyter, 1926. 194 pp.

This book is a presentation from the view point of formal logic, of the fundamental operations and theorems of algebra. Chapter one discusses the usual axioms of algebra and the immediately resulting theorems. Articles ten and eleven of this chapter deal in particular with natural numbers. The next four chapters define the usual algebraic operations for point sets, vectors and matrices, and discuss the validity of the axioms of Chapter one in the fields of these elements. It should not be surprising that these axioms do not all hold. For, although two axioms may be alike in form, they are different axioms if they deal with different elements, unless of course they only state relations between properties common to both sets of elements. For example the statement, p. 19, that A_2 failing and I_4 holding for point-sets implies the independence of these axioms, has no significance. This merely shows that the point-set analogs of A_2 and I_4 are inconsistent. Notwithstanding this criticism, the reviewer considers these chapters a worthwhile parallelling of these various algebras.

Chapters six to ten inclusive treat in a brief and attractive way the theory of linear equations, linear vector forms, linear and quadratic forms, and matrices and determinants. Chapter eleven gives a brief axiomatic introduction to the theory of finite groups. The last chapter discusses the genetic structure of algebra upon the basis of Peano's five axioms. The treatment is condensed, well organized and very readable.

GLENN JAMES

Beyond the Electron. By Sir J. J. Thomson. Cambridge University Press, 1928. 43 pp.

The contents of this highly interesting booklet is a lecture given at Girton College and includes mathematical appendices bearing on points involved. The incentive was obtained from the experiments of the author's son, Professor G. P. Thomson, showing that a moving electron is always accompanied by a series of waves; a result also obtained from the experiments of Davisson and Germer. Because of these experiments the old idea that an electron is merely a charged particle must be given up. In his characteristically lucid manner, Sir Joseph goes step by step in forming pictures of two physical things: the transmission of energy and the propagation of waves and presents ultimately a view of the electron with a dual structure, "one part of this structure, that where the energy is located, being built up of a number of lines of electric force, while the other part is a train of waves in resonance with the electron and which determine the path along which it travels."

In view of the attempts of physicists to reconcile phenomena explainable in some cases by the undulatory theory and in other cases by the corpuscular conceptions of the physical entities: light, X-rays, electrons, etc., the booklet should prove of great interest to mathematicians and physicists alike.

A. F. KOVARIK

Aufgabensammlung zur Funktionentheorie. By Konrad Knopp. Volume II: *Aufgaben zur höheren Funktionentheorie.* Sammlung Göschen, No. 878. Berlin and Leipzig, Walter de Gruyter, 1928. 143 pp.

This little book is the continuation of Volume I (*Aufgaben zur elementaren Funktionentheorie*, No. 877) which appeared under the same title in 1923. Like all other books published by Dr. Knopp, large as well as small ones, the present volume is a new contribution, small, but significant, to the widely known reputation of its author as an excellent scholar and talented teacher. The difficult task of selecting from the immense material of the modern theory of functions the problems just within the reach of a beginner is here masterfully accomplished. A student who will go faithfully through 183 problems of Vol. I and 187 problems of Vol. II will almost imperceptibly find himself in possession of a technique which is absolutely necessary for starting an advanced study; at the same time, he will acquire general information as to the present situation of the subject. He will find problems (of different grades of difficulty) on power- and Laurent-series, on singular points of analytic functions and calculus of residues, on factorization and partial fraction expansion of entire and meromorphic functions, on simple and double periodic functions, on analytic continuation and behavior of power series on their circles of convergence, on Riemann surfaces and conformal transformations. The solutions which follow the problems are condensed enough to be valuable even to students who use them. It is the reviewer's opinion that the two volumes of the *Aufgabensammlung* together with the two volumes of Knopp's *Funktionentheorie* (Sammlung Göschen, Nos. 668, 703) can be successfully used in this country in teaching the first course of the theory of functions of a complex variable.

J. D. TAMARKIN

L'Évolution des Idées Géométriques dans la Pensée Grecque: Point, Ligne, Surface. By Federigo Enriques. Translated by Maurice Solovine. Paris, Gauthier-Villars, 1927.

This short treatise contains in outline the history of ancient geometry as a basis for an exposition of the philosophy of Greek geometry and the relation of ancient to modern thought. The merit of this tract lies in the exhibition of the philosophic movement. The historic part reveals certain omissions which would have necessitated the re-phrasing of some parts of the discussion relating to the evolution of the logic of geometry. Thus, in outlining Egyptian geometry, the author fails to point to the fact which has become evident in recent years that early Egyptian writers had advanced further than was formerly supposed and had reached results which as far as Greek testimony indicates, were not transmitted to ancient Greece. Thus the Greeks did not know of the remarkable approximation to the area of the circle, contained in the Rhind papyrus. More striking yet, is the content of another Egyptian papyrus (the Moscow papyrus) presumably of equal antiquity, which, like a flash of lightning in the darkness, momentarily illumines the obscure past. It contains the correct computation of the volume of the frustrum of a square pyramid. Had Enriques had in mind

this startling achievement, his discussion of the origin of the theorem relating to the volume of a pyramid would have been somewhat different. His discussion of the critique of Pythagorean concepts of points due to Parmenides and Zeno of Elea is excellent, as is also his account of the development of the infinitesimal analysis of Democritus and Archimedes, and his comparison of it with modern views. There is another historical point to which it may be worth while to direct attention. Bertrand Russell and A. N. Whitehead refer to Weierstrass as the first to banish the fixed infinitesimal from the differential and integral calculus. Enriques mentions Weierstrass, but also Cauchy and Dini. We wish to remind the reader of the historical fact that the fixed infinitesimal was banished, in the works of Benjamin Robins in 1735, of Colin Maclaurin in 1742, and of Simon Lhuilier in 1786, though, of course, these men had not reached the arithmetization of the theory of limits of the time of Weierstrass.

FLORIAN CAJORI

Invariants of Quadratic Differential Forms. By O. Veblen. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 24.) Cambridge University Press, 1927. viii+102 pp.

This book has been prepared to replace Tract No. 9 of the same series, *Invariants of Quadratic Differential Forms*, by J. E. Wright, which has been out of print for a number of years.

As pointed out in the preface, the presentation is formal in character as the space would permit of only the simplest applications. In Chapter I certain preliminaries are taken up such as the notation, the Kronecker deltas and then application to theorems on determinants. Chapter II is entitled *Differential Invariants*. Here the author considers such concepts as *n-dimensional space, coordinate system, invariant, tensor*, in a manner which commends the highest praise. Nowhere else have I seen these ideas so carefully presented. Chapter III considers quadratic differential forms and the theory of covariant differentiation and the curvature tensor. Chapter IV is devoted to euclidean geometry. The development is carried through largely for an n-dimensional space and in terms of a coordinate system not assumed to be Cartesian. In this fashion the expressions for a number of the metrical invariants of Riemannian geometry are introduced. Chapter V is given to a study of the problem of Christoffel concerning the equivalence of two quadratic differential forms. Chapter VI deals principally with the geometry of paths through the powerful method of "normal coordinates." A short historical section appears at the end of each chapter which serves to point out the chief sources of material.

In the opinion of the reviewer this book is exceedingly well done. The careful formulation of the underlying concepts and the meticulous phrasing of the definitions make this little book an invaluable one to a beginner in the subject, and for the same reason, it seems to me it must afford the adept a high degree of aesthetic enjoyment.

J. H. TAYLOR

NOTES

At a meeting held in New York City on December 27, 1928, a new scientific society, the Acoustical Society of America, was formed to bring together workers in all branches of pure and applied acoustics. Dr. Harvey Fletcher, of the Bell Telephone Laboratories, was elected president.

The Vienna Academy of Sciences has awarded its Lieben prize, for the best mathematical work by an Austrian during the preceding three years, to Professor Karl Menger, for his memoirs on the theory of dimensions.

The Accademia Pontificia dei Nuovi Lincei has announced that it will award a prize of 10,000 lire for an unpublished critical work on the theory of quanta. Competing manuscripts, which may be in English, French, German, Spanish, Italian, or Latin, should reach the Academy by October 30, 1929. Competition is open to all except members of the Academy.

The Royal Academy of Naples has awarded its biennial prize to Professor Enea Bortolotti, for his memoir entitled *Geometria degli spazi riemanniani*.

The gold medal of the Royal Astronomical Society has been awarded to Professor Ejnar Hertzsprung, of the Leyden Observatory, for his work in stellar astronomy.

The Royal Society of Edinburgh has awarded its Gunning Victoria Jubilee prize for the period of 1924–28 to Professor E. T. Whittaker, for his contributions to mathematical science and the promotion of mathematical research in Scotland.

Professors A. A. Michelson, of the University of Chicago, and R. A. Millikan, of the California Institute of Technology, were awarded gold medals by the Society of Arts and Sciences at a meeting in New York City, February 22, 1929.

The American Institute of Electrical Engineers has made the following awards: the Edison gold medal for achievement in electrical science to Dr. F. B. Jewett, for his pioneer research and development work in connection with the theory and practice of voice transmission; the Lamme medal to Mr. Allan Bertram Field "for the mathematical and experimental investigation of eddy current losses in large slot-wound conductors in electrical machinery."

Professor Aurel Stodola, of the Zurich Technical School, has been elected correspondent of the Paris Academy of Sciences in the section of mechanics.

Professors Oskar Perron and Heinrich Tietze, of the University of Munich, have received the title of Geheimer Regierungsrat.

Professor F. Hausdorff, of the University of Bonn, has been elected a member of the Moscow Mathematical Society.

The Council of the Royal Society of London has recommended for election as fellow Professor G. R. Goldsbrough, of Armstrong College, Newcastle-on-Tyne.

Mr. William Bowie, of the United States Coast and Geodetic Survey, has been elected correspondent of the Paris Academy of Sciences in the section of geography and navigation.

Dr. W. J. Humphreys, of the United States Weather Bureau, has been elected a corresponding member of the Meteorological Society of Hungary.

The University of Oxford has conferred the degree of doctor of science on Professor Oswald Veblen, of Princeton University.

Associate Professor Leopold Vietoris, of the University of Innsbruck, has been appointed professor of mathematics at the Vienna Technical School.

Professor Léon Brioullin, of the Collège de France, has been appointed to the newly established professorship of physical theories at the Sorbonne.

Dr. Maurice Fréchet, maître de conférences at the Sorbonne, has been appointed "professeur sans chair" by the Faculty of Sciences of the University of Paris.

Dr. Edwin Feyer, of the Breslau Technical School, has been promoted to an associate professorship.

Associate Professor R. Furch, of the University of Rostock, has been promoted to a professorship.

Dr. Ferenz Jüttner has been appointed to an associate professorship of theoretical physics at the University of Freiburg i. Br.

Associate Professor Wolfgang Krull, of the University of Freiburg i. Br., has been appointed professor of mathematics at the University of Erlangen.

Dr. Karl Loewner, of the University of Berlin, has been appointed an associate professor of mathematics at the University of Cologne.

Dr. B. L. van der Waerden, of the University of Göttingen, has been appointed professor of mathematics at the University of Groningen.

Professor Enea Bortolotti has been appointed professor of analytic projective geometry at the University of Cagliari.

Professor Luighi Brusotti, of the University of Cagliari, has been transferred to the professorship of analytic geometry at the University of Pisa.

Professor G. Juvet, of the University of Neuchâtel, has been appointed professor at the University of Lausanne, as successor to Professor L. Maillard, retired.

Professor R. Nevanlinna, of the University of Helsingfors, lectured at the Zurich Technical School during the year 1928–29, in place of Professor Hermann Weyl, on leave of absence in the United States.

The following have been admitted as docents: Dr. F. Löbell, at the Stuttgart Technical School; Dr. Robert Remak, at the University of Berlin.

The following graduate courses in mathematics are announced for the summer of 1929.

UNIVERSITY OF CALIFORNIA, BERKELEY, intersession, May 20–June 29.—By Professor T. M. Putnam: Theory of algebraic equations and of infinite series, Geometric introduction to the theory of functions.—By Professor D. N. Lehmer: Metric differential geometry. Summer session, July 1–August 10: Professor B. A. Bernstein: Elementary algebra for advanced students.—Professor E. R. Hedrick: Analytic geometry of space, Functions of a complex variable.—Professor James Pierpont: Non-euclidean geometry.—Professor Hermann Weyl: Representations of groups, applications and representations of groups to quantum physics.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES, July 1–August 10.—Professor Harriet E. Glazier: Foundations of arithmetic.—Professor M. W. Haskell: Advanced geometry, Geometric introduction to the theory of functions.—Professor Sophia H. Levy: The Teaching of mathematics.

UNIVERSITY OF CHICAGO, first term, June 17–July 24; second term, July 25–August 30.—By Professor H. E. Slaught: Differential equations, Elliptic integrals.—By Professor A. C. Lunn: Relativity, Lattices and crystal groups.—By Professor A. B. Coble (Illinois): Analytic projective geometry, Algebraic geometry.—By Professor L. M. Graves: Introduction to analysis, Functions of lines.—By Professor R. W. Barnard: Theory of equations, Metric differential geometry.—By Professor W. Bartky: Theoretical mechanics, Modern theories of analytic differential equations.—By Professor C. C. MacDuffee (Ohio State): Theory of numbers, Theory of algebraic numbers, Linear algebras.—By Professor H. S. Everett: Advanced calculus.

UNIVERSITY OF COLORADO, first term, June 18 to July 21; second term, July 23 to August 24.—By Professor Light: Teacher's course in mathematics; History of mathematics; Partial differential equations.—By Professor Kempner: Algebraic analysis; Projective geometry; Differential equations. Second term.—By Professor Light: Statistics; Teacher's course (repeated); Partial differential equations (continued). —By Professor Kempner: Projective geometry (continued); Differential equations (continued).—By Professor Kendall: Theory of equations.

COLUMBIA UNIVERSITY, July 8 to August 16.—By Professor G. D. Birkhoff: Mathematical elements of art; Introduction to relativity.—

By Professor W. B. Fite: Differential equations.—By Professor J. F. Ritt: Theory of numbers.—By Professor B. O. Koopman: Functions of a real variable.

CORNELL UNIVERSITY, July 6 to August 16.—By Professor Virgil Snyder: Teachers' course; Projective geometry.—By Professor F. R. Sharpe: Advanced analytic geometry.—By Professor W. A. Hurwitz: Advanced calculus.—By Professor W. B. Carver: Theory of numbers. —By Professor C. F. Craig: Elementary differential equations. Reading and research will be directed by Professors J. I. Hutchinson, Snyder, Sharpe, Hurwitz, Carver, D. C. Gillespie, Craig, and C. F. Roos.

UNIVERSITY OF ILLINOIS, June 17 to August 10.—By Professor G. A. Miller: The theory of numbers.—By Professor R. D. Carmichael: Partial differential equations.—By Professor E. B. Lytle: Teachers' course; Theory of equations and determinants.—By Professor H. Levy: Geometric transformations.—By Dr. V. A. Hoersch: Advanced calculus.—By Dr. F. C. Ogg: Projective geometry.

UNIVERSITY OF INDIANA, June 13 to August 9.—By Professor S. C. Davisson: Theory of functions of a complex variable; Differential equations; Theory of equations.—By Professor D. A. Rothrock: Partial differential equations; Advanced calculus.—By Professor H. E. Wolfe: College geometry; Analytic mechanics.—By Professor C. B. Hennel: General mathematics; Analytic geometry.

UNIVERSITY OF IOWA, first term, June 8 to July 19.—By Dr. M. A. Nordgaard: Subject matter and teaching of mathematics.—By Dr. Conkwright: Ordinary differential equations; Theory of numbers.—By Professor Wylie: Celestial mechanics; Mathematics of finance; Descriptive astronomy.—By Professor Ward: Modern geometry.—By Professor Chittenden: Advanced calculus; Orthogonal functions.—By Professor Rietz; Actuarial theory and practice; Statistics. Second term, July 22 to August 23.—By Mr. McCoy: Matrices and determinants.—By Dr. Nordgaard: The history of mathematics.—By Professor Ward: Modern geometry; Differential equations.—By Professor Reilly: Algebra for high school teachers; Linear difference equations.

JOHNS HOPKINS UNIVERSITY, July 1 to August 9.—By Dr. John Williamson: Theory of functions of a complex variable.

UNIVERSITY OF KANSAS, first term, June 12 to July 20.—By Professor C. H. Ashton: Advanced calculus; seminar.—By Professor U. G. Mitchell: Projective geometry I; Teachers' course in mathematics; seminar. Second term, July 22 to August 17.—By Professor Mitchell: Projective geometry II; History of mathematics; seminar.

UNIVERSITY OF MICHIGAN, June 24 to August 16.—By Professor J. W. Bradshaw: Higher algebra; Projective geometry.—By Professor P. Field: Vector analysis; Applied mathematics, engineering problems.—By Professor W. B. Ford: Advanced calculus; Infinite series with special reference

to Fourier series.—By Professor T. H. Hildebrandt: Theory of functions of a real variable; Partial differential equations.—By Professor L. C. Karpinski: Teaching of geometry; History of mathematics.—By Professor T. R. Running: Empirical formulas.—By Professor H. C. Carver: Advanced mathematical theory of statistics.—By Professor L. A. Hopkins: Analytic mechanics; Celestial mechanics.—By Professor N. H. Anning: Differential equations.—By Professor C. J. Coe: Integral equations.—By Professor J. A. Nyswander: Theory of probability; Finite differences.—By Professor R. L. Wilder: Foundations of mathematics.—By Mr. N. C. Fisk: Graphical methods.—By Mr. D. K. Kazarinoff: Aerodynamics.

UNIVERSITY OF MINNESOTA, first term, June 18 to July 27.—By Professor Dunham Jackson: History of ancient and modern mathematics.—By Professor Elizabeth Carlson: Differential equations.—By Professorial Lecturer J. V. Uspensky: Theory of numbers.—By Professor Jackson: Fourier, Legendre and Bessel series.—By Professors Raymond Brink, Dunham Jackson, and Professorial Lecturer J. V. Uspensky: Reading in advanced mathematics. Second term, July 29 to August 31.—By Professorial Lecturer J. V. Uspensky: Recent developments in the mathematical theory of probability.—By Professor A. L. Underhill and Professorial Lecturer J. V. Uspensky: Reading in advanced mathematics.

OHIO STATE UNIVERSITY, June 18 to August 30.—By Professor S. E. Rasor: The teaching of mathematics; Theory of functions of a complex variable; Advanced calculus.—By Professor A. D. Michal: Continuous groups; Tensor analysis.—By Professor Grace Bareis: Projective geometry.

UNIVERSITY OF PENNSYLVANIA, July 1 to August 10.—By Professor H. H. Mitchell: Galois theory of equations.—By Professor J. R. Kline: Functions of a complex variable.—By Professor F. D. Murnaghan, of Johns Hopkins University: Inversive geometry; Linear differential equations.—By Professor J. M. Thomas: Integral invariants.

UNIVERSITY OF PITTSBURGH.—By Professor K. D. Swartzel: Functions of a complex variable; Teaching of mathematics.—By Professor F. A. Foraker: Modern synthetic geometry; Solid analytic geometry.—By Associate Professor Taylor: Advanced calculus; Functions of a real variable.—By Assistant Professor Culver: Theory of equations.

STANFORD UNIVERSITY, June 20 to August 31.—By Professor W. A. Manning: Group theory; Theory of functions.—By Professor G. T. Whyburn (Texas): Point-set theory.

UNIVERSITY OF TEXAS, first term, June 4 to July 15.—By Professor R. L. Moore: Functions of real variables; Foundations of geometry.—By Professor E. L. Dodd: Probability; Analytic functions.—By Professor H. J. Ettlinger: Differential equations; Ruler and compass constructions.—By Professor H. S. Vandiver: Number theory; Advanced calculus.—By Professor A. E. Cooper: Advanced calculus; Theory of equations.—By Professor P. M. Batchelder: Teaching problems in mathematics. Second term, July 15 to August 26.—By Professor R. L. Moore: Functions of real vari-

ables (continued); Non-euclidean geometry.—By Professor H. J. Ettlinger: Partial differential equations; Definite integrals.—By Professor E. G. Keller: Advanced calculus.

UNIVERSITY OF WISCONSIN, July 1 to August 9.—By Professor R. W. Babcock: Vector analysis.—By Professor H. W. March: Differential equations; Definite integrals.—By Professor E. B. Skinner: Differential geometry; Finite groups; Infinite series. Special nine weeks session for graduates, July 1 to August 30.—By Professor M. H. Ingraham: Higher algebra; Theory of approximations.—By Professor Warren Weaver: Theory of relativity; Advanced electrodynamics; Complex variable theory. (Only one of the two last named courses will be given, the choice depending upon the demand.)—By Professor J. H. Van Vleck of the Department of Physics: Introduction to atomic theory and line spectra; Dielectric and magnetic media; Quantum mechanics and chemistry. (Only one of the two last named courses will be given, the choice depending upon the demand.)

The following forty-nine doctorates with mathematics or mathematical physics as major subject were conferred during 1928; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

A. A. Albert, Chicago, August, *Algebras and their radicals and division algebras.*

H. E. Arnold, Yale, June, *The rational space quintic curve of the second species and its relation to the rational plane quartic curve.*

May M. Beenken, Chicago, June, *Surfaces in five-dimensional space.*

T. C. Benton, Pennsylvania, June, *On continuous curves which are homogeneous except for a finite number of points.*

A. H. Blue, Iowa, July, *On the structure of sets of points of classes one, two, and three.*

G. B. Briggs, Princeton, June, *On types of knotted curves.*

P'ei Yuan Chou, California Institute of Technology, June, theoretical physics, *The gravitational field of a body with rotational symmetry in Einstein's theory of gravitation.*

L. W. Cohen, Michigan, June, *On subsets of separable metric space homeomorphic with subsets of the linear continuum.*

H. A. Davis, Cornell, June, physics, *Involutorial transformations belonging to a linear complex.*

H. A. DoBell, Cornell, February, industrial organization, *On the geometry of the triangle.*

D. C. Duncan, California, May, astronomy, *Rational quintic curves autopolar with respect to a finite number of conics.*

J. M. Earl, Minnesota, July, physics, *Polynomials of best approximation on an infinite interval.*

J. J. Gergen, Rice, June, *Generalized lacunae, On Taylor's series admitting the circle of convergence as a cut, On accessible points on the boundary of a three dimensional region.*

A. O. Hickson, Chicago, August, *An application of the calculus of variations to boundary-value problems.*

E. L. Hill, Minnesota, June, major physics, minor mathematics, *Quantum mechanics of the rotational distortion of spin multiplets in molecular spectra.*

Rosa L. Jackson, Chicago, August, *The boundary-value problem of the second variation for parametric problems in the calculus of variations.*

R. L. Jeffery, Cornell, June, physics, *The sequences of functions which define a definite integral containing a parameter.*

Marie M. Johnson, Chicago, August, *Tensors of the calculus of variations.*

B. W. Jones, Chicago, June, *Representation by positive ternary quadratic orms.*

E. G. Keller, Chicago, August, *On the origin of a planet from a ring system.*

G. H. Keuligan, Johns Hopkins, June, *Vibrations of an elongated* U*-bar.*

M. S. Knebelman, Princeton, June, *Collineations and motions in generalized spaces.*

Mark Kormes, Columbia, January, *On basis sets.*

Lincoln LaPaz, Chicago, August, *An inverse problem of the calculus of variations.*

W. T. MacCreadie, Cornell, February, physics, *On the stability of the motion of a viscous fluid.*

Morris Marden, Harvard, June, *On the location of the roots of the jacobian of two binary forms and of the derivative of a rational function.*

W. L. Moore, Illinois, June, mathematical physics, *On the geometry of the Weddle surface.*

D. C. Morrow, Chicago, June, *The determination of all quaternary quadratic forms which represent every positive integer.*

F. W. Perkins, Harvard, February, *On the oscillation of harmonic functions.*

J. W. Peters, Johns Hopkins, June, *Invariants of sets of points under inversion.*

O. J. Peterson, Michigan, June, *On the rational plane quintic with three cusps.*

C. G. Phipps, Minnesota, July, physics, *Problems in approximation by functions of given continuity.*

Allie W. Richeson, Johns Hopkins, June, *Pentagons inscribed in circles.*

W. C. Risselman, Minnesota, July, physics, *Approximation to a given function by means of polynomials in another given function.*

V. B. Rojansky, Minnesota, August, major physics, minor mathematics, *The Stark effect of hydrogenic atoms in the new quantum mechanics.*

W. E. Roth, Wisconsin, June, applied mathematics, *A solution of the matrix equation* $P(X)=A$.

C. A. Rupp, Chicago, June, *An extension of Pascal's theorem to space of r dimensions.*

N. E. Rutt, Pennsylvania, June, *Concerning the cut points of a continuous curve when the arc curve contains exactly n independent arcs.*

S. A. Schelkunoff, Columbia, May, *On certain properties of the metrical and generalized metrical groups in linear spaces of n dimensions.*

A. A. Shaghoian, California, May, analytical mechanics, *Solution of homogeneous linear difference equations by means of infinite determinants.*

C. D. Smith, Iowa, February, *On generalized Tchebycheff inequalities in mathematical statistics.*

F. E. Smith, Catholic, June, physics and philosophy, *The triangles in- and circumscribed to the triangular symmetric rational quartic curve.*

Dan Sun, Chicago, August, *Projective differential geometry of quadruples of surfaces with points in correspondence.*

E. L. Thompson, Chicago, June, *Systems of two differential equations from the Lie group standpoint.*

Morgan Ward, California Institute of Technology, June, physics, *The foundation of general arithmetic.*

Marie J. Weiss, Stanford, June, *Primitive groups which contain substitutions of prime order p and of degree 6p or 7p.*

C. O. Williamson, Chicago, August, *Stability of an airplane with rotating propeller.*

D. W. Woodard, Pennsylvania, June, *Two dimensional analysis situs, with special reference to the Jordan curve-theorem.*

Ko-Chuen Yang, Chicago, August, *Various generalizations of Waring's problem.*

Princeton University has received from its Class of '87 the sum of $200,000 to endow the chair of astronomy held by Professor H. N. Russell, a member of that class. The class also presented a portrait of Professor Russell to the University.

Professor G. H. Hardy lectured at the University of Iowa, March 23, 1929, on *Hilbert's Logic*, and at Columbia University, April 17–18, on *Prime Numbers*.

Professor and Mrs. E. B. Van Vleck will sail from San Francisco on August 15, for a trip around the world. Professor Van Vleck will retire from active teaching in June, 1929.

Professor Hermann Weyl, of the Zurich Technical School, who was appointed Visiting Lecturer for this Society for the present year, has delivered lectures recently at Harvard University (April 29–May 3), Chicago (May 7), Illinois (May 8–10), The Rice Institute (May 20–22), California Institute of Technology (May 31–June 3), Stanford University (June 7–10). The subjects of his lectures have been: *Gravitation and the electron*, *Consistency of mathematics*, *The spherical symmetry of atoms*, *The theory of groups and quantum mechanics*, and *The symmetric and the linear group*.

Dr. Gregory Breit, mathematical physicist of the Department of Terrestrial Magnetism, has been assigned for a year to work in atomic physics in various laboratories and universities of Europe.

Dr. Elizabeth B. Cowley, of Vassar College, has been promoted to a professorship of mathematics.

Professor Werner Heisenberg, of the University of Leipzig, will lecture at the University of Chicago on theoretical physics, during the summer quarter.

Dr. A. O. Hickson, of Brown University, has been appointed assistant professor of mathematics at Duke University.

Dr. M. S. Knebelman has been appointed assistant professor of mathematics at Princeton University.

Mr. E. R. C. Miles, of Rice Institute, has been appointed assistant professor of mathematics at Duke University.

Professor C. H. Richardson, formerly of Georgetown College, now professor of mathematics at Bucknell University, was recently elected by the Faculty of Bucknell University to serve as head of the department of mathematics for the current academic year.

Dr. C. H. Smiley of the University of Illinois, has been appointed Guggenheim Fellow for 1929–30.

Dr. W. H. Taylor has been appointed head of the department of mathematics at Alabama College.

Associate Professor Robert Torrey, of the University of Mississippi, was granted leave of absence for the first semester of the present academic year. Professor Torrey was injured in a train wreck last June.

Professor G. T. Whyburn, recently appointed at Johns Hopkins, has been appointed Guggenheim Fellow for next year, and will have leave of absence from Johns Hopkins.

Professor J. Boussinesq, member of the section of mechanics of the Paris Academy of Sciences, died February 19, 1929, at the age of eighty-six.

Professor Luighi Carlini, of the Technical School of Udine, died December 28, 1928.

Mr. A. K. Erlang, of the Copenhagen Telephone Company, known for his work in the applications of mathematics to telephone problems, died February 3, 1929, at the age of fifty-one.

Professor Johannes von Kries, of Freiburg i. Br., the distinguished physiologist, editor of the German edition of Helmholtz's *Physiological Optics* and author of *Die Prinzipien der Wahrscheinlichkeitsrechnung*, died December 30, 1928, at the age of seventy-five.

Professor Samuel Oppenheim, of the department of astronomy of the University of Vienna, died August 15, 1928, at the age of seventy.

Professor P. H. Puiseux, honorary astronomer of the Paris Observatory, died September 28, 1928.

Professor Charles Riquier died at Caen in January, 1929, at the age of seventy-eight.

Professor Giovanni Russo, of the Technical School at Catanzaro, died June 4, 1928, at the age of seventy-seven.

Mr. W. B. Alcock, senior fellow of Emmanuel College, Cambridge, died January 18, 1929, at the age of seventy-one.

Professor A. W. Bickerton, of Canterbury College, Christchurch, New Zealand, known for his impact theory of cosmic evolution, died January 23, 1929, at the age of eighty-seven.

Mr. T. H. Blakesley, honorary secretary of the London Physical Society, known for his work in the theory of alternating currents, died February 13, 1929, at the age of eighty-one.

Mr. M. J. M. Hill, of University College, London, died January 11, 1929, at the age of seventy-two.

Mr. A. W. Prater, of the University of California at Los Angeles, died April 16, 1929.

Professor W. H. Sherk, of the University of Buffalo, died in January, 1929.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

AUERBACH (F.). Lebendige Mathematik. Breslau, Hirt, 1928.

BARRAU (J. A.). Figuren en hare coordinaten. Groningen, Wolters, 1928. 19 pp.

VAN DEN BERG (F.). De multi-sektie van den hoek in de vlakke meetkunde op 5 verschillende wijzen aangetoond. Rotterdam, the author, 1927.

BLASCHKE (W.). Vorlesungen über Differentialgeometrie. Band 3. Berlin, Springer, 1929.

BREUER (S.). See EULER (L.).

BROWN (O. E.). See NAZINOFF (P. S.).

BURNSIDE (W. S.) and PANTON (A. W.). The theory of equations with an introduction to the theory of binary algebraic forms. Edited by M. W. J. Fry. 7th edition. Volume 2. London, Longmans, 1928.

CAJORI (F.). A history of mathematical notations. Volume 2: Notations mainly in higher mathematics. Chicago, Open Court, 1929. 18+367 pp.

COURANT (R.). Vorlesungen über Differential-und Integralrechnung. Band 2: Funktionen mehrerer Veränderlicher. Berlin, Springer, 1929. 360 pp.

VAN DIEREN (C. M.). Nulstelsels in het platte vlak. (Dissertation, Utrecht.) Utrecht, Drukkerij Zuidam, 1928. 77 pp.

VAN DITMARSH (A. J.). Afbeelding van de lijnelementen van een vlak op den complex der Raaklijnen van een monoïde. (Dissertation, Utrecht.) Zutphen, Nauta, 1928. 29 pp.

DÜSING (K.). Leitfaden der Kurvenlehre. Mit zahlreichen Anwendungen aus der Technik von E. Preger. 3te vermehrte Auflage, bearbeitet von E. Wilde. Leipzig, Jänecke, 1928. 8+143 pp.

EULER (L.). Drei Abhandlungen über die Auflösung der Gleichungen. (1738, 1764, 1790.) Übersetzt und herausgegeben von S. Breuer. (Ostwalds Klassiker der Exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1928. 94 pp.

EUWE (W.). Differentiaal-invarianten en partiëelle differentiaalvergelijkingen uit de tensorrekening. (Dissertation, Amsterdam.) Amsterdam, H. J. Paris, 1928. 12+84 pp.

FORD (W. B.). College algebra. Revised edition. New York, Macmillan, 1928. 6+278 pp.

FRY (M. W. J.). See BURNSIDE (W. S.).

HAAG (J.). Cours complet de mathématiques élémentaires. Tome 3. Paris, Gauthier-Villars, 1928. 346 pp.

HARDING (A. M.) and MULLINS (G. W.). Plane trigonometry. New York, Macmillan, 1928. 8+118 pp.

HASSE (H.) und SCHOLTZ (H.). Die Grundlagenkrisis der griechischen Mathematik. Charlottenburg, Panverlag Kurt Mentzner, 1928. 72 pp.

HOESTRA (H. P.). Onderzoek van eenige bilineaire congruenties van kubische ruimtekrommen door eene afbeelding op het puntenfeld. (Dissertation, Utrecht.) Utrecht, van Druten, 1928. 80 pp.

HOHEISEL (G.). Partielle Differentialgleichungen. (Sammlung Göschen.) Berlin, de Gruyter, 1928. 159 pp.

VON HOLZHAUSEN (W.). Logik und Zweckreinheit im neudeutschen Schachproblem. Berlin, de Gruyter, 1928. 114 pp.

KÜHNE (O.). Die mathematische Schule in der Nationalökonomie. Band I, 1ter Teil: Die italienische Schule bis 1924. Berlin, de Gruyter, 1928.

LENNES (N. J.). College algebra. New York, Harper, 1928. 14+301 pp.

LIETZMANN (W.). Aus der Mathematik der Alten. Leipzig, Teubner, 1928.

LÖFFLER (E.). Ziffern und Ziffernsystem. Teil 1. Leipzig, Teubner, 1928. 2+54 pp.

LORIA (G.). Storia delle matematiche. Volume I: Antichita, medio evo, rin ascimento. Torino, S.T.E.N., 1929. 497 pp.

MULLINS (G. W.). See HARDING (A. M.).

NAZIMOFF (P. S.). Applications of the theory of elliptic functions to the theory of numbers. Translated from the Russian by A. E. Ross. Mimeographed by O. E. Brown. Evanston, 1928. 146 pp. (On sale at the University of Chicago Book Store: $3.00.)

OSGOOD (W. F.). Lehrbuch der Funktionentheorie. 5te Auflage, 1ter Band, und 2te Auflage, 2ter Band, 1te Lieferung. Leipzig, Teubner, 1928, 1929. 14+818+8+307 pp.

PANTON (A. W.). See BURNSIDE (W. S.).

PESCI (G.). Lezioni di calcolo numerico, grafico e meccanico. Livorno, Autografia della Reale Accademia Navale, 1928.

PREGER (E.). See DÜSING (K.).

ROSS (A. E.). See NAZIMOFF (P. S.).

ROTHE (R.). Höhere Mathematik für Mathematiker, Physiker und Ingenieure. Teil II. Leipzig, Teubner, 1929. 8+201 pp.

RUTGERS (J. G.). See SCHUH (F.).

SCHMEHL (C.). Die Jacobische Erzeugungsweise der Flächen zweiten Grades. Giessen Roth, 1929.

SCHOLZ (H.). See HASSE (H.).

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STAMMLER (G.). Begriff, Urteil, Schluss. Untersuchungen über Grundlagen und Aufbau der Logik. Halle a. S., Niemeyer, 1928.

STRACHAN (R. C.). A table of hyperbolic radians, adapted to the derivation of hyperbolic functions from trigonometric tables. New York, the author, 1928.

STRAUCH (J.). Die primitiven Wurzeln im allgemeinen Zahlkörper

(Dissertation, Leipzig.) Leipzig, Druckerei der Werkgemeinschaft, 1927. 11 pp.

SWETT (G. W.). Construction of alignment charts. New York, Wiley, 1928. 98 pp.

THOMPSON (A. J.). Logarithmetica Brittanica. Being a standard table of logarithms to twenty decimal places. Part 4: Numbers 40,000 to 50,000. (Tracts for Computers, No. 16.) Cambridge, University Press, 1929. 6+100 pp.

VIVANTI (G.). Elemente der Theorie der linearen Integralgleichungen. Übersetzt und mit Anmerkungen versehen von F. Schwank. Hannover, Helwingsche Verlagsbuchhandlung, 1929.

WILDE (E.). See DÜSING (K.).

PART II. APPLIED MATHEMATICS

ANGENHEISTER (G.). See GEIGER (H.).

BACON (R.). Opera hactenus inedita. Fasciculus VIII: Questiones supra libros quatuor phisicorum Aristotelis, nunc primum edidit F. N. Delorme collaborante Robert Steele. Oxford, Clarendon Press, 1928. 22+284 pp.

—— The Opus Majus of Roger Bacon. A translation by R. B. Burke. 2 volumes. Philadelphia, University of Pennsylvania Press, and London, Oxford University Press, 1928. 13+418+6+422 pp.

BARKHAUSEN (H.). Elektronen-Röhren. 3te Auflage. Band 2: Röhrensender. Leipzig, Hirzel, 1928. 6+121 pp.

BEAVER (J. L.). Elements of alternating currents and alternating current apparatus. 2d edition. New York, Longmans, 1928.

BOSLER (J.). Cours d'astronomie, Faculté des Sciences de Paris. Tome 3: Astrophysique. Paris, Hermann, 1928. 5+723 pp.

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BUSEMANN (A.). See GEIGER (H.).

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DAHL (O. G. C.). Electric circuits. Theory and applications. Volume 1: Short calculations and steady-state theory. New York, McGraw-Hill, 1928. 10+390 pp.

DEBYE (P.). See SOMMERFELD (A.).

DELORME (F. N.). See BACON (R.).

ENRERA (J.). Polarisation diélectrique. Paris, Les Presses Universitaires de France, 1928. 172 pp.

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FORSTERLING (K.). Lehrbuch der Optik. Leipzig, Hirzel, 1928. 12+610 pp.

FOWLER (R. H.). Statistical mechanics. The theory of the properties of matter in equilibrium. Cambridge, University Press, 1929. 570 pp.

GEIGER (H.) und SCHEEL (K.), herausgegeben von. Handbuch der Physik. Band 6: Mechanik der elastischen Körper. Bearbeitet von G. Angenheister, A. Busemann und O. Föppl. Redigiert von R. Grammel. Berlin, Springer, 1928. 12+632 pp.

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GIORGI (G.). Che cos è elettricità? Roma, Paolo Cremonese, 1928. 136 pp.

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HOCART (R.). Problèmes et calculs de chimie générale. Paris, Gauthier-Villars, 1928. 182 pp.

IMMLER (W.). Flugzeugnavigation. München, Oldenbourg, 1928.

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JORDAN (H. H.) and PORTER (F. M.). Descriptive geometry. Boston, Ginn, 1929. 12+349 pp.

KIRSCH (E.). See PETTERSSON (H.).

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—— Probleme der modernen Physik, Arnold Sommerfeld zum 60. Geburtstag gewidmet von seinen Schülern. Herausgegeben von P. Debye. Leipzig, Hirzel, 1928. 221 pp.

STEELE (R.). See BACON (R.).

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THOMSON (G. P.). See THOMSON (J. J.).

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